TEACHING LOCUS AT UNDERGRADUATE LEVEL: A CREATIVITY APPROACH

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In this paper, we present our experience, while working in the MC Squared project, with the design and evaluation of educational digital resources aiming at promoting creative mathematical thinking among undergraduate students. The resources, called "c-books" (c for creative), are produced within an innovative socio-technological environment by a community gathering together computer scientists and researchers in mathematics and mathematics education. The paper highlights the importance of teaching loci curves at undergraduate *level as an introduction to implicit equations. It presents the design choices providing the c-book with affordances to promote creativity in mathematics in terms of personalized non-linear path, constructivist approach, and meta-cognition based activities, among others. The paper also presents experts' a priori evaluation of the c-book mathematical creativity potential.*

Keywords: Creative mathematical thinking, socio-technology environment, locus, implicit equations, experimental geometry.

INTRODUCTION

Historically, mathematical curves essentially occurred as loci, e.g. a parabola as the locus of all points having the same distance from a given fixed line and a given fixed point outside the line. The concept of a curve within a coordinate system was only quite recently developed by Descartes, Fermat, and others in the 17th century – about 2000 years after the Greek had performed quite detailed studies of various curves, and also had created interesting and important new curves for specific purposes, purely via their description as loci (Boyer, 2004, pp. 74-102).

The curves in Descartes' "La Geometrie" (Descartes, 1637, see also Boyer, 2004, pp. 74-102) then arise naturally as implicit curves as a result of solving systems of implicit equations where each equation represents a condition on the geometric objects involved. In the following centuries, curves and in particular the special cases of graphs of functions in one variable were being studied deeply. But at the beginning of the 20th century when Felix Klein was working a lot on the question of how to teach mathematics, implicit equations still played a very important role for him. E.g., in the first section on algebra in his Mathematics from a Higher Standpoint (Klein, 1924, part I, pp. 93-109) when he discusses simple examples such as graphs of quadratic functions, he immediately uses implicit equations as well, namely some discriminants describing the reality of the roots of the functions. It was only later that some others misinterpreted his ideas to focus on the importance of functions in a too narrow way, namely only to functions from R to R.

Besides the historical importance of loci and their often implicit equations there are many reasons for using them at school level. The following is an important one: The description of a curve as a locus gives a more intrinsic description and also a more operational description than an equation. So, even for curves appearing as graphs of functions, looking at them as geometric loci often deepens their understanding. Moreover, using loci and implicit equations early in teaching is a good preparation for studying implicit equations later in linear algebra, e.g. planes and the classification of quadrics. It is also a good companion to the implicit equation of a circle which is otherwise a quite isolated example of such an equation in many cases (sometimes, ellipses are also mentioned, at least in their standard form $x^2/a^2+y^2/b^2=1$, but the fact that implicitly described curves are a most natural thing to consider is often not mentioned).

The importance of implicit loci arises even more at the undergraduate level: for example, a curve might be associated not only with an explicit equation, a function graph, a parametrized curve, or an implicit algebraic equation but as well with solutions of differential equations. It is important to understand function graphs, implicit equations and parametrized curves as loci in order to be able to fully grasp differential geometry tools such as tangent line or plane, curvature, osculating circles or ellipsoids. In real life mathematics or engineering, most objects are loci of some sort. Control theory for example deals with trying to keep a mobile position not far from a target trajectory, with the help of integral and differential calculus. The investigation of soft loci with a dynamic geometry system (Healy, 2000; Laborde, 2005) is very helpful in building this picture in the mind of students.

The flexible production of loci is of paramount interest in industry and design to define curves and surfaces used in computer aided design, such as Bézier curves and their variants (see Piegl, 2013). The main feature of those is the fact that they can be described in many different ways: as parametrized curves, as implicit curves, and also as loci. All those descriptions have their advantages for the application at hand such as: Through the parameterization, many properties of the curves can be studied easily; with the implicit equation, it is straightforward to decide if a given point is on the curve, or on one side or the other side; and the description as a locus provides a numerically robust and quick way to compute points on the curve, draw it or 3D-print it.

Promoting creative mathematical thinking (CMT) is a central aim of the European Union by being connected to personal and social empowerment for future citizens (EC, 2006). It is also considered as a highly valued asset in industry and as a prerequisite for meeting economic challenges. Exploratory digital media provide users with potential for developing CMT in unprecedented ways (Hoyles & Noss, 2003; Healy & Kynigos, 2010). Yet, new designs are needed to support learners' engagement with CMT in collectives using dynamic digital media.

The MC Squared project, briefly presented in the next section, looks for new methodologies that would assist designers of digital educational media to explore, identify and bring forth resources stimulating more creative ways of mathematical thinking. The paper focuses on the design of one such resource, the “Experimental Geometry” c-book, highlighting the design choices and the resource affordances to foster CMT in its users. Concluding remarks bringing forward factors stimulating creativity in the collaborative design of digital educational resources are proposed in the final section. The research work presented in this paper is related to (Trgalova, El-Demerdash, Labs, &Nicaud, 2016), submitted to ICME13, with more emphasis on the locus theoretical background and its importance to introduce implicit equations at the undergraduate level. Design choices to foster CMT are as well detailed here.

THE MC SQUARED PROJECT

MC Squared project (http://mc2-project.eu/) aims at designing a software system, the “c-book environment”, to support stakeholders from creative industries producing educational content to engage in collective forms of creative design of appropriate digital media. The c-book environment provides an authorable tool including diverse dynamic widgets, an authorable data analytics engine and a tool supporting collaborative design of resources called “c-books”. The project studies the processes of collaborative design of c-books intended to enhance CMT.

CREATIVE MATHEMATICAL THINKING

Based on a literature review studying creativity (El-Demerdash, 2010; El-Demerdash & Kortenkamp, 2009; Haylock 1997; Weth 1998, among others), CMT has been defined in the project as an intellectual activity generating new mathematical ideas in a non-routine mathematical situation. Drawing on Guilford’s (1950) model, the generation of new ideas shows fluency (the ability to generate quantities of ideas), flexibility (the ability to create different categories of ideas), originality (the ability to generate new and unique ideas that others are not likely to generate), and elaboration (the ability to redefine a problem to create others by changing one or more aspects).

THE "EXPERIMENTAL GEOMETRY" C-BOOK

The notion of geometric loci of points is the topic of the “Experimental Geometry” c-book presented in this section. According to Jare and Pech (2013), this notion is difficult to grasp and technology can be an appropriate media to facilitate its learning. The authors suggest one way is to use dynamic geometry software to “find the searched locus and state a conjecture” and a computer algebra system to “identify the locus equation”.

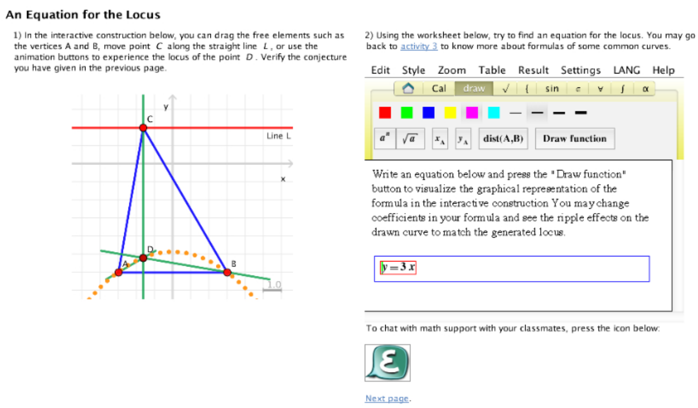

Fig. 1: A screenshot of a c-book page showing three widgets: Cinderella, EpsilonWriter and EpsilonChat.

The challenge in designing this c-book was to exploit c-book technology affordances to propose a comprehensive study of geometric and algebraic characterization of some loci within the c-book. We decided to create activities aiming at studying loci of important points in a triangle. These loci (for example locus of the orthocenter) are generated by the movement of one vertex of a triangle along a line parallel to the opposite side (see Fig. 1). These are classical problems from the field of geometry of movement that were proposed for teaching purposes even before the advent of dynamic geometry (Botsch, 1956). Elschenbroich (2001) revisits the problem of locus of the orthocenter in a triangle with a new media, dynamic geometry software. El-Demerdash (2010) uses this example to promote CMT among mathematically gifted students at high schools.

The c-book invites students to experiment geometric loci generated by intersection points of special lines of a triangle while one of its vertices moves along a line parallel to the opposite side (see Fig. 2b). The activity can give rise to a number of various configurations, which makes it a rich situation for exploring, conjecturing, experimenting, and proving.

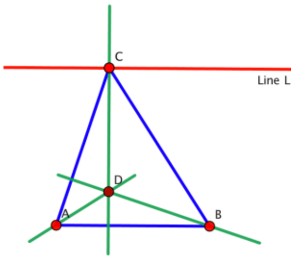

Fig. 2a: Geometrical situation proposed with Cinderella (Act. 1, page 1).

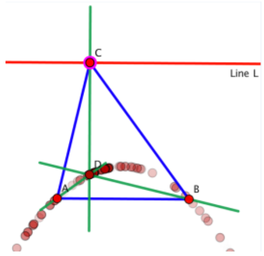

Fig. 2b: Visualizing the trace of D while C moves on the red line (Act. 1, page 2).

The c-book is organized in three sections to which the student can refer in case he needs or wants it. The first section proposes the main activity called "Loci of special points of a triangle". It starts by inviting the students to explore, with Cinderella (http://www.cinderella.de) dynamic geometry software, the geometric locus of the orthocenter of a triangle while one of its vertices moves along a line parallel to the opposite side (Fig. 2a). The students are asked to explore the situation, formulate a conjecture about the geometrical locus of the point D (page 1), and test the conjecture (page 2) by visualizing the trace of the point D (Fig. 2b).

On page 3, the students are asked to find an algebraic formula of the locus, which is a parabola. The formula is to be written with the EpsilonWriter software (http://epsilonwriter.com/en/) and the interoperability between this widget and Cinderella allows the students to check whether the provided formula fits the locus or not.

The next pages invite the students to think of, explore, and experiment the geometrical loci in other similar situations, such as the locus of the circumcenter (intersection of the perpendicular bisectors), the incenter (intersection of the angle bisectors) or the centroid (intersection of the medians). Other situations can be generated by considering the intersection of two different lines, for example a height and a perpendicular bisector. Twelve such situations can be generated. For each case, one page is devoted offering to the students:

a **b**

Fig. 3: Circle as a locus of points that are at a given distance from a given point: (a) "soft" locus, and (b) "robust" locus.

- a Cinderella widget with a triangle ABC such that the vertex C moves along a line parallel to [AB] and a collection of tools for constructing intersection point, midpoint, line, perpendicular line, angle bisector, locus, as well as the tool for visualizing the trace of a point;
- an EpsilonWriter widget enabling a communication with Cinderella;
- EpsilonChat widget enabling remote communication among students.

The second section called “The concept of geometric locus” introduces the concept of locus of points. It starts by a baby example, leading the students to “discover” the fact that a circle can be characterized as a locus of points that are at the same distance from a given point (page 1). The students first experiment a “soft” locus (Healy, 2000; Laborde, 2005) of a point A placed at the distance 6 cm from a given point M (Fig. 3a), and then they verify their conjectures by observing a “robust” construction of the circle centered at A with a radius 6 cm (Fig. 3b).

The next page is constructed in a similar way in order to allow the students to explore perpendicular bisector as a geometric locus of points that are at a same distance from two given points. Finally, the page 3 proposes a synthesis of these two activities and provides a definition of the concept of geometrical locus of points.

The third activity, “Algebraic representation of loci”, proposes a guided discovery of algebraic characterization of the main curves that can be generated as loci of points as those in section 2.

Design choices and rationale

Personalized non-linear path: The c-book is designed to allow students to go through it according to their knowledge and interest. They are invited to enter by the main activity (section 1). However, the concept of geometric locus is a prerequisite. In case this knowledge is not acquired yet, or the students need revising it, they can reach the section 2 by an internal hyperlink from various places of the main activity. Similarly, section 3, which allows the students to develop knowledge about the algebraic characterization of some common curves that is useful in the main activity, is reachable from the main activity. Thus the students can “read” the c-book autonomously, in a non-linear personalized way, deepening their knowledge about geometric or algebraic aspects of loci of points according to their needs.

Promoting creative mathematical thinking: The c-book is designed in a way to support the development of creative mathematical thinking through promoting its four components (fluency, flexibility, originality, and elaboration) among undergraduate students. First, the main activity is designed in a way to call for students’ elaboration: they are invited to modify the initial situation by considering various combinations of special lines in a triangle, whose intersection point generates a locus to explore. Fluency and flexibility are fostered by providing the students a rich environment in which they can explore geometric configurations and related algebraic formulas while benefitting from a feedback allowing them to control their actions and verify their conjectures (see feedback and learning analytics section). Specific feedback is implemented toward directing students to produce different and varied situations and help them to break down their mind fixation by considering yet different configurations, such as two different kinds of special lines in a triangle passing through the movable vertex (e.g. a height intersecting with an angle bisector), and then the intersection of two different lines that do not pass through the movable vertex. The c-book provides the students not only with digital tools enabling them to explore geometric and algebraic aspects of the studied loci separately, but also with a so-called “cross-widget communication” affordances between Cinderella, a dynamic geometry environment, and EpsilonWriter, a dynamic algebra environment, which makes it possible to experimentally discover the algebraic formula that matches the generated locus in a unique way; this feature may contribute to the development of original approaches by the students.

Constructivist approach – Learning by doing – Guided discovery: The c-book activities in both section 2 and section 3 are developed based on the constructivist learning theory practices through guided discovery approach in order to enable students to create new experiences and link them to their prior cognitive structure supported with learning opportunities for conjecturing, exploration, explanation, and mathematics communication.

Meta-cognition - Learning by reflecting and promoting mathematics communication skills: All c-book sections end up with a meta-cognitive activity that has been designed to encourage students to reflect about their learning and enable them further understanding, analysis, and control of their cognitive processes. These activities have been also designed aimed at the development of students' written mathematical communication skills through the use of EpsilonChat mathematical chat engine.

Technological development: An outstanding feature of the c-book environment is the fact that it does not only come with a large number of existing widgets in the mathematical context from several different European developer teams, but it also comes with so-called widget factories, one from each of the developer teams allowing authors to generate their own specialized widgets, if they want. The interesting point of this is that all these diverse widgets work perfectly together with the back-end of the environment and they can even collaborate with each other within pages. For example, the dynamic algebra system EpsilonWriter is an interesting tool for manipulating formulas and equations via a unique drag and drop interface (right part of Fig. 1). But it neither has a built-in function graphing tool nor geometric construction capabilities. These aspects are some of the specialties of the programmable dynamic geometry system Cinderella (left part of Fig. 1).

Later, when working with the c-book, a student may have produced a reasonable equation for a function within EpsilonWriter, and she can visualize it by using the 'draw' tab. The graph of the function will be shown in the Cinderella construction at the right. For the student, this is visually clear and intuitive; but technically a lot is happening in the background: First, the equation will be sent from the EpsilonWriter software via a standardized protocol to the c-book environment and from there to the Cinderella software which finally visualizes it as a part of the interactive construction. All this is possible within the c-book player running in a web-browser.

As the example above illustrates, cross-widget communication is a quite powerful feature. In this case, it opens the opportunity for the c-book author to make explicit connections between different representations of a mathematical object: a curve represented as a geometrical locus, its formula or equation with the ability to modify it dynamically, and a geometric figure combining both the construction as a locus and the visualization of the curve given by the equation. Within the c-book environment, such opportunities exist in other branches of mathematics as well: e.g., via this mechanism statistics and probability widgets may be connected to geometry, algebra, a number theory widget or even to a logo programming widget, to name just a few more use cases.

Learning analytics and feedback: Another advantage of the c-book environment and the widget factories working with it is that it is easy for a c-book is to decide which of the student's actions should be logged to a database while she is studying the c-book. There have been many different types of logs implemented in this c-book that enable the teacher to capture the student's path in studying the c-book. Two types of feedback are provided to students, while they are studying the c-book to guarantee their smooth move from page to page and switch between the c-book activities: technical feedback and mathematical or educational feedback for CMT, breaking down mind fixations.

A PRIORI EVALUATION OF THE C-BOOK CMT AFFORDANCES

The c-book CMT affordances were evaluated by "experts" (researchers involved in the MC Squared project). This a priori evaluation was guided by the following two research questions:

R.Q1 - Which of the four cognitive components of CMT: fluency, flexibility, originality and elaboration, and social and affective aspects have been better integrated and promoted through the design of the c-book units? That is, what affordances are perceived by the evaluators as enhancers of these components?

R.Q2 - Is there any correlation among the cognitive components of CMT, as perceived by the evaluators?

In order to answer the above research questions we used an evaluation tool called "CMT Affordances Grid" (Appendix A). This tool was developed and refined within the MC Squared project. The grid contains three sections. The 13 first items evaluate the c-book affordances towards the development of mathematical creativity in users/students. These items address the c-book affordances such as nature of the activities or variety of representations of mathematical concepts at stake and ask the evaluators to what extent these affordances are likely to enhance the user's cognitive processes (fluency, flexibility, originality, elaboration). The second and third sections deal with social and affective aspects of the c-book that are likely to enhance mathematical creativity in its users.

As for the first aspect, the responders were asked to evaluate the items in relation to each one of the four cognitive components of mathematical creativity in a scale from 1 (weak affordance) up to 4 (strong affordance). There was an extra option called N/A in case the affordance was not applicable for the specific item.

The evaluation of the mathematical creativity affordances of this c-book was done by three experts in the field of mathematics education, a senior researcher, a post-doctoral researcher, and a Ph. D. student who were not involved in its design. It was organized in three steps. First, the evaluators had to use the c-book and be acquainted with the affordances. Second, a teleconference was organized by the main designer of the c-book to address evaluators' needs for understanding and clarification. Third, the evaluators evaluated the c-book affordances based on the grid using an online form prepared for this purpose.

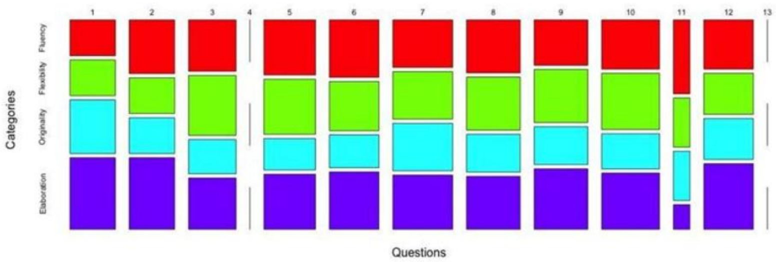

Fig. 4: Evaluation of CMT Cognitive from the experts' point of view on CMT affordances Grid.

The chart, shown in Fig. 4, represents the evaluation of the cognitive components of CMT from the experts' point of view. The height of the bars represents the mean value of each component (fluency, flexibility, originality and elaboration), while the thickness represents the mean between the four aspects for each question. From the evaluators' point of view, there are no affordances on the items 4 and 13, which means that the c-book does not establish connections between different knowledge areas and mathematics (item 4) and it does not include half-baked constructs that call for intervention (item 13). On the other hand, the evaluators consider that the c-book encourages exploratory activity and user experimentations (item 7) and encourages also generalizing mathematical phenomena, going from concrete cases to general ones or generalizing real world phenomena through the use of mathematics (item 10).

In Table 1, we present the quantitative data for each component computing the mean from No Affordance (scored 1) to Strong Affordance (scored 4). From the scale defined to evaluate the c-book we got the following values for each component, as shown in the Table 1 above: Fluency = 2.53, Flexibility = 2.46, Originality = 1.96 and Elaboration = 2.92. Except the originality component, all other components are in the range of “weak to possible” affordances. The originality got a value of 1.96 which means “no affordance”. However, the value is quite close to “weak” affordance.

Fluency	Flexibility	Originality	Elaboration	Social	Affective
2.53	2.46	1.96	2.92	2.3	1.6

Table 1. CMT Evaluation Summary

The highest value for this c-book in terms of cognitive aspects was elaboration for which the value achieved the rank of "good affordance". It means that, in general, the c-book is judged to have a potential to boost the students’ development of their ability to provide many responses or to come up with many strategies to solve a mathematical problem or challenge. Fluency and flexibility are the components with lower values of good affordance.

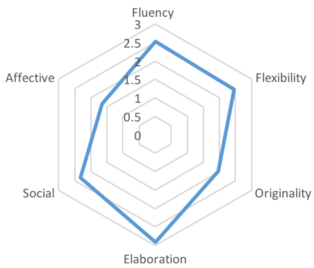

Fig. 5: Radar Distribution of CMT aspects.

The radar chart (Fig. 5) shows the distribution of the evaluation among the evaluated categories. This chart shows which component of CMT is most likely to be enhanced by the use of the c-book. In the case of this c-book it is the elaboration aspect, followed by fluency and flexibility.

	Fluency	Flexibility	Originality	Elaboration
Fluency	1.00	0.94	0.83	0.89
Flexibility	0.94	1.00	0.82	0.84
Originality	0.83	0.82	1.00	0.92
Elaboration	0.89	0.84	0.92	1.00

Table 2. Correlation Values of CMT Components

Table 2, collating the 13 questionnaire items, shows correlations among the four cognitive components of CMT. We can notice that the correlations are strong between some cognitive aspects. It means that, considering a significant value of $r > 0.80$ ($p = 0.05$), we may conclude that fluency, flexibility and elaboration can be fostered at the same time. In the case of originality, there is no statistical evidence that supports the hypothesis that this component can be fostered by the other ones.

We can conclude that even though the c-book main activity is designed to call for students' elaboration (they are invited to modify the initial situation by considering various combinations of special lines in a triangle, whose intersection point generates a locus to explore), fluency and flexibility are fostered by providing the students a rich environment in which they can explore geometric situations and try out algebraic formulas whereas benefitting from a feedback system allowing them to control their actions and verify their conjectures. Specific feedback is implemented toward directing students to produce different and varied situations and help them to break down their mind fixation by considering yet different configurations.

The c-book provides the students not only with digital tools enabling them to explore geometric and algebraic aspects of the studied loci separately, but also with a so-called "cross-widget communication" of Cinderella and EpsilonWriter, which makes it possible to experimentally discover the algebraic formula that matches the generated locus in a unique way; this feature may contribute to the development of original approaches by the students.

CONCLUSION

The c-book presented in this paper is the result of a collaborative work of a group of designers coming from various professional backgrounds, as the group comprises researchers in mathematics, mathematics education and computer science, as well as educational software developers. Without the synergy among those group members, a number of design choices would have remained in a hypothetical state, namely the technological advances in terms of cross-widget communication and learning analytics features. The design of the c-book has thus become a driving force in the c-book technology development, and in return, the unique c-book technology features enabled the creation of a resource with affordances promoting creative mathematical thinking.

This experience brings to the fore factors stimulating creativity in the collaborative design of digital educational resources. Among these are the following two:

- A variety of designers' profiles, as pointed out by Fischer (2005), as it encourages the search for novel information and perspectives;
- A close collaboration with software developers which is critical for the design and implementation of unique features of the c-book technology resulting in a creative resource. Thus the development of the technology and the educational resources designed with this technology feeds each other.

REFERENCES

Botsch, O. (1956). Bewegungsgeometrie. Reinhardt-Zeisberg, Band 4b. Moritz Diesterweg.

Boyer, C. B. (2012). History of analytic geometry. Courier Corporation Publications.

Descartes, R. (1637). La Géométrie. Appendix to Discours de la méthode.

EC (2006). Recommendation 2006/962/EC of the European Parliament and of the Council of 18 December 2006 on key competences for lifelong learning. Official Journal of the European Union, L 394, 10–18.

El-Demerdash, M. (2010). The Effectiveness of an Enrichment Program Using Dynamic Geometry Software in Developing Mathematically Gifted Students' Geometric Creativity in High Schools. Doctoral dissertation, University of Education Schwaebisch Gmünd, Germany.

El-Demerdash, M. & Kortenkamp, U. (2009). The effectiveness of an enrichment program using dynamic geometry software in developing mathematically gifted students' geometric creativity. Paper Presented at the 9th International Conference on Technology in Mathematics Teaching – ICTMT 9, Metz, France, July 6-9, 2009.

Elschenbroich, H. J. (2001). Dem Hoehenschnittpunk auf der Spur. In Herget, W. et al; (Eds.), Medien verbreiten Mathematik (Proceedings) (pp. 86-91), Berlin: Verlag Franzbecker, Hildesheim.

Guilford, J. P. (1950). Creativity. American Psychologist, 5, 444–454.

Fischer, G. (2005). Distances and diversity: sources for social creativity. In Proceedings of the 5th conference on Creativity & cognition (pp. 128-136), ACM New York, NY, USA.

Guilford, J. P. (1950). Creativity. American Psychologist, 5, 444–454.

Haylock, D. (1997). Recognizing mathematical creativity in schoolchildren. Zentralblatt für Didaktik der Mathematik, 27(2), 68-74.

Healy, L. (2000) Identifying and explaining geometrical relationship: interactions with robust and soft Cabri constructions In: Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education, T. Nakahara and M. Koyama (Eds.) (Vol.1, pp. 103-117) Hiroshima: Hiroshima University

Healy, L., & Kynigos, C. (2010). Charting the microworld territory over time: design and construction in learning, teaching and developing mathematics. Zentralblatt für Didaktik der Mathematik, 42(1), 63-76.

Hoyles, C., & Noss, R. (2003). What can digital technologies take from and bring to research in mathematics education? In A. J. Bishop et al. (Eds.), Second International Handbook of Mathematics Education. Dordrecht: Kluwer Academic Publishers

Trgalova, J., El-Demerdash, M., Labs, O., & Nicaud, J. (2016). Collaborative design of educational digital resources for promoting creative mathematical thinking. Paper Presented at the 13th International Congress on Mathematical Education – ICME 13, Hamburg, Germany, July 24-31, 2016

Jares, J., & Pech, P. (2013). Exploring loci of points by DGS and CAS in teaching geometry. Electronic Journal of Mathematics and Technology, 7(2), 143-154.

Klein, F. (1924). Elementarmathematik vom h?heren Standpunkte aus, Band I. Dritte Auflage. Springer.

Laborde, C. (2005). Robust and soft constructions: two sides of the use of dynamic geometry environments. In Proceedings of the 10th Asian Technology Conference in Mathematics, Korea National University of Education, pp. 22-35.

Peigl, L. A., & Wayne, T. (2013). The NURBS book: monographs in visual communication. Springer, New York.

APPENDICES

Appendix A: CMT Affordances Grid