EXPLOITING POTENTIALS OF DYNAMIC REPRESENTATIONS OF FUNCTIONS WITH PARALLEL AXES

Giulia Lisarelli

University of Florence, Department of Mathematics; giulia.lisarelli@unifi.it

The concept of function has a central role both at school and in everyday situations. Several studies revealed that it is hard for students to think of functions and graphs in terms of covariation and this could contribute to their struggles in Calculus. The emergence of available technologies has fostered new teaching and learning approaches to overcome students' difficulties and some of them concerns the use of dynamic algebra and geometry software programs to experience the dependence relation and to explore functions as covariation. In this paper we describe a particular representation of functions with parallel axes and the analysis of a protocol in which four students work together on a problem that involves the exploration of a function represented in a dynamic interactive file. The analysis has been carried out to explore the potential of the proposed dynamic representation of functions that incorporates the semantic domain of space, time and movement.

Keywords: dynamic algebra and geometry software, dragging, function.

INTRODUCTION

The concept of function is very important both in secondary school and university mathematics but it also has a central role in everyday situations. For a long time, this notion has been at the core of several studies in mathematics education, and a rich literature has revealed students' difficulties in understanding the concept in all its aspects (Vinner & Dreyfus, 1989; Tall, 1991; Dubinsky & Harel, 1992). Difficulties in interpreting the dependence relation as a dynamic relation between covarying quantities are widely reported (Goldenberg et al., 1992; Carlson et al., 2002) and also difficulties in manipulating graphs and recognizing functions' properties from graphs (Carlson & Oehrtman, 2005).

Indeed, the tendency to think of functions and graphs as static objects, rather than as dynamic processes, may contribute to students' struggles in the learning of Calculus (Ng, 2016). At the same time the emergence of a variety of new available software has fostered new teaching and learning approaches. Therefore, we can find several studies about the use of technology to manipulate multiple representations of functions (Healy & Sinclair, 2007; Sinclair et al., 2009).

Falcade et al. (2007) suggest that the use of a dynamic algebra and geometry software, such as GeoGebra, allows students to experience functions as covariation, that is a crucial aspect of the idea of function (Confrey & Smith, 1995; Tall, 1996). According to these assumptions we are interested in studying students' cognitive processes involved in working with functions represented in a dynamic environment.

In this paper we describe a particular dynamic representation of functions and some results from a pilot study conducted last year. This study is part of a larger research project whose focus is investigating how certain aspects of the mathematical concept of function could be supported by such dynamic representation. Moreover, we are interested in exploring the semiotic potential of the representation of functions with parallel axes (Bartolini Bussi & Mariotti, 2008), to gain insight into how to exploit it didactically.

DYNAGRAPHS

Dynagraphs, as they have been referred to by Goldenberg et al. (1992), are particular representations of functions obtained by using a dynamic software, which consist in representing both the x- and y-axes horizontally, in one dimension, unlike the Cartesian graphs which represent functions in two dimensions. The underlying assumption is that this kind of representation can support a dynamic conception of functions because it draws attention to variables' variations and movements and to the relation between these variations.

We now propose a description of our development of this idea, implemented within the algebra and geometry software GeoGebra. In particular, we designed a sequence of activities aimed at making the representation of functions in the Cartesian plane rich in meanings. We start with a kind of dynagraph and evolve its design, through a sequence of activities, in order to reach the Cartesian graph.

As we can see in Figure 1 the first dynagraph has one horizontal line, with 0 and 1 marked, and two little ticks that can move on it in this way: one of them represents the independent variable and can always be dragged, the other one represents the dependent variable, it cannot be directly dragged, but it moves depending on the movements of the independent tick. We note that the variables are represented by ticks and not by points, because a point is usually seen as a pair of coordinates, while a tick better expresses the idea of "value". Moreover, there are two points marked on the line that determine the unit segment, to highlight that it is the real number line.

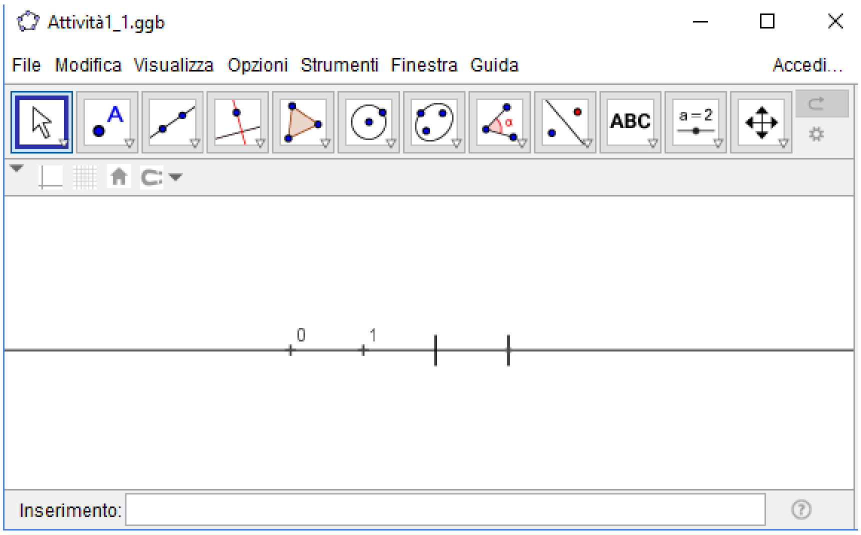

Figure 1. Dynagraph

The design of our representations also allows to separate out the two variables, that is, to create a copy of the line in order to have one notch on each line. So what can be seen on the screen changes because there is a fixed horizontal line, representing the x-axis, and its double, representing the y-axis, that can be dragged up and down maintaining the parallelism, and the alignment of the origin.

Thanks to the dragging and the design of the dynamic files there is the opportunity to rotate the y-axis, joining the zeros and making it orthogonal to the x-axis; obtaining a representation that includes the Cartesian axes on which two ticks can move. As described above, the tick on the x-axis can be directly dragged while the other one moves depending on it. The following step for the construction of the Cartesian graph of a function consists of the construction of the point (x, f(x)) and, finally, by activating the trace tool on this point and dragging the independent variable we obtain the graph, as showed in Figure 2. In the rest of this paper we will only discuss activities with parallel axes, investigating their semiotic and didactic potential.

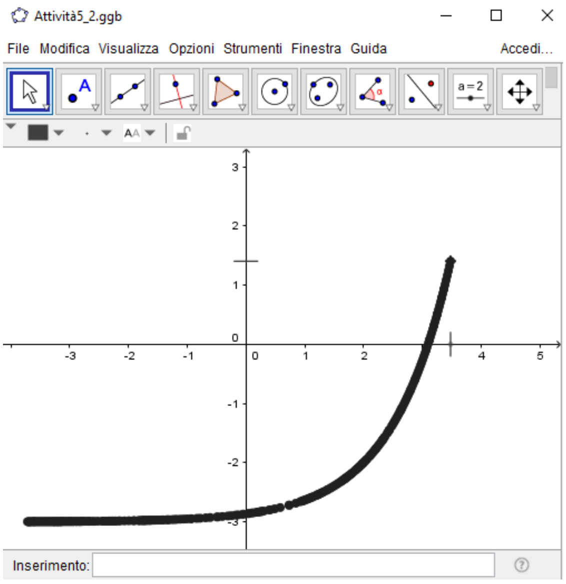

Figure 2. Cartesian graph

PILOT STUDY

As we mentioned above, a first experimentation was conducted last year in a 10th grade of an Italian High School for Math and Science, where we introduced students to the function concept through dynagraphs. Students worked in pairs on pre-designed dynamic interactive files that they were asked to explore. The tasks proposed in these files were open, in order to support students' explorations, and working in pairs was to foster their speaking aloud and explaining their reasoning to each other. Lessons were video-recorded through two cameras.

The foundational goal of the pilot study was to build the mathematical meaning of functional dependence, as a relation between two covarying quantities: one depending on the other one. We expected to start from the relation between the movement of the two ticks bounded to the lines.

Starting from the representation of function on one horizontal line, we designed a sequence of activities that led to the Cartesian graph of functions, following a trajectory like the one described in the previous section. We proposed several examples, including not everywhere defined functions and discontinuous functions, in order to support the production of situated signs related to the mathematical concepts of domain, limit, continuity and asymptote. We also expected that this kind of one-dimensional representation would foster the description of relative movements of the ticks and comparisons between possible walks followed by the ticks on the lines. For example, students could recognize symmetry or concordant movements that we would identify as situated signs for monotonicity's properties of functions. Speaking about advanced mathematical concepts, we also expected that a description of change in speed could be read mathematically as an attention to the slope of the function, that is its derivative.

Our choice to let the user decide to see two distinct lines or to have them overlap is led by the observation that we think there could be some cases for which it is convenient to have two separated axes (for example to explore functions like $f(x) = \sqrt{x}$ or $g(x) = |x|$) and some other cases in which it is easier to work with one line with two variables moving along it (for example to determine $f(x) = x$).

Analysis of an activity

The Theory of semiotic mediation (Bartolini Bussi & Mariotti, 2008) describes the semiotic potential of an artifact as follows:

> On the one hand, personal meanings are related to the use of the artifact, in particular in relation to the aim of accomplishing the task; on the other hand, mathematical meanings may be related to the artifact and its use. This double semiotic relationship is named the *semiotic potential of an artifact*.

According to this definition we analyze the semiotic potential of the representation of functions with parallel axes focusing on the embedded knowledge and the utilizations schemes that students employ when exploring the dynamic files.

In the first lesson of the pilot study, after the exploration of a linear function, students are asked to explore the dynagraph of the function $f(x) = \frac{1}{x-4}$ and to write down their observations. We chose to give them this not everywhere defined function in order to introduce them to the mathematical concept of domain, while exploring the dependence relation between the two variables. We expected that the particular behaviour of the dependent variable in a neighborhood of the vertical asymptote $x = 4$ could have supported the employment of a language referring to the movement and to the relation between the movements of the two ticks. Moreover we expected that students would have noticed the existence of the horizontal asymptote $y = 0$ in terms of changing in speed of the tick representing the dependent variable.

In the next sections we analyze some excerpts from this lesson with the goal of recognizing instances in which the semiotic potential of dynagraphs seems to be exploited.

Excerpt 1

The following transcript is a dialogue between four students who are interacting with a GeoGebra file that represents the dynagraph of the function $\frac{1}{x-4}$. We chose this excerpt from the first lesson because in it students frequently use words that refer to variables' movements and to the relation between these movements. Moreover, as we expected, the function's behaviour in a neighborhood of the vertical asymptote $x = 4$ causes students' astonishment and some interesting observations.

1	Gian:	Oh no, it is going crazy!
2	Fra:	Look there, it dashes backwards
3	Gian:	It makes certain leaps!
4	Fra:	Ah, but are they three points here?
5	Dar:	What? Here there is back to the future!
6	Fra:	Eh eh, there are three points guys
7	Rob:	No
8	Fra:	Or not?
9	Gian:	This one doesn't move, and the meeting point is the same, it doesn't change.
10	Fra:	No no they are two, indeed I tried to make some changes but they are equal, actually they are the same

As we can read from the dialogue, the discontinuity of the function is something very interesting for these students, because when they drag the independent variable they see the dependent one disappear from one side of the screen and then re-appear from the other side of the screen. They try to interpret this phenomenon by using a “continuous” interpretation. Fra supposes that there could be three points, possibly because he does not accept that one point can run off on one side and come back from the other side. But another interesting fact is that he modifies the tick representing the dependent variable in order to convince himself that the points are two and not three: dragging the independent tick he always sees the same output. Therefore the feedback is directly given by the software, and by useful manipulations made on the file.

Let us now look at a sentence that we consider as a first sign, situated in the context of the dynamic file, of the mathematical concept of domain of the function. It is important to observe that the representation of the function with parallel axes requires the following interpretation of the domain: this needs to be read on the y-axis because the independent variable can always be dragged, bounded to its line. So we could say that a point on the x-axis belongs to the domain of the function if it has a corresponding output on the y-axis.

As we can read from Rob’s words there are different aspects of the semiotic potential of the representation of the function that come to light. Indeed, he refers to time (after a moment), to space (upper, below) and to movement (a range of movement).

24	Rob:	After a moment, the upper point moves only in a certain range of movement of the point below.

The next excerpt concerns a description of the asymptotic behavior of the function when x tends to infinity.

Excerpt 2

71	Fra:	But do you see how it dashes away? Look!
72	Dar:	Try to move a bit further backwards, look, it still moves very little.
73	Fra:	It continues to move
74	Dar:	Do you see? it moves a little bit
75	Fra:	Yes, it is moving a little bit
76	Dar:	Look, it moves here
77	Rob:	Nothing is moving, where do you see that it moves?
78	Fra:	It moves you’re right, yes
79	Rob:	No, here it does not move
80	Fra:	Yes Rob it moves, look!
81	Rob:	Zoom in zoom in, so we can see it. And then it makes certain leaps…
82	Fra:	It leaps it leaps!
83	Rob:	Look, it has leapt to one side
84	Fra:	And then it stops
85	Rob:	That’s it, from here on it is fixed, look

Recalling the verb (to dash away) used previously as well, Fra underlines the unexpected acceleration of the dependent variable (71). Then the other students observe that $f(x)$ makes some leaps (81) when dragging the x in a neighborhood of the point where the function is not defined. The semiotic potential of this dynagraph comes into play in the mathematical concept of derivative:

by dragging the x-tick in a neighborhood of $x = 4$ the $f(x)$-tick leaps, which corresponds to a function having a very high slope.

Then students discuss about the function's behavior for x tending to negative infinity. Dar suggests that $f(x)$ still moves when x is dragged backwards (72), that is x tending towards negative infinity; and Fra agrees (82). But Rob prefers to zoom in because it seems to him the x-tick to be fixed and he would convince himself of the contrary. Again the semiotic potential of the dynagraph comes into play, supporting with respect to the mathematical concept of limit; aspects of such potential can be observed in students' words and actions. In particular, zooming in students can observe the function's behavior for ever smaller variations of the independent variable.

In the last sentence (85) Rob refers to x values bigger than zero and far from it, and we are sure of it because we see from the video that he is dragging the x-tick to the right on its line.

Before the discussion we would just notice that the analysis of the two excerpts reveals some interesting considerations consistent with the *a priori* analysis of the designed activity.

DISCUSSION

The studies on the interaction between humans, technology and mathematics have to take into account a variety of aspects: the relation between the teacher and the technology in the mediation of mathematical knowledge, or how this knowledge is influenced by constraints and actions allowed in the technological environment, and several other components that are involved. In this paper we have presented a study to better understand the explorations of functional dependence in a dynamic algebra and geometry environment. In particular, we have analysed aspects of the semiotic potential of the representation of functions with parallel axes, presenting some excerpts from a pilot study conducted last year.

We noticed that students' descriptions of dynagraphs are rich in references to movement, time and space. We think that it could be fostered by the dynamic environment, by the task that requires for the exploration, and by the possibility of dragging. However such richness could also be affected by the fact that the students never met the concept of function (in high school) before, so they have not yet developed a formal mathematical vocabulary about functions, so the use of these terms becomes necessary for them.

From the analyses we can infer that introducing students to functions through dynagraphs seems to promote a covariational view of functions, seen as relations between the movements of quantities that are varying in an interval of real numbers. In the same way also some mathematical properties of functions are conceived dynamically, for example Rob identifies the domain of the function as a certain range of movement of the independent variable. Consistent with our expectation about this kind of one-dimensional representation, that it would have fostered the description of relative movements of the ticks and comparisons between possible walks followed by the ticks on the lines, we also notice students frequent use of verbs strictly related to movement and speed (2, 3, 9, Excerpt 2).

Finally, we highlight students' creativity, revealed by their use of the tools offered by the software, for example Fra changes the $f(x)$-tick's visualization and Rob zooms in to convince himself that $f(x)$ moves on.

In a future study it could be interesting to investigate whether students' conceptions of functions evolve, and if so how. In particular we are interested in analyzing students' use of references to movement and time when they are taught the mathematical definitions: do they disappear or do they last? How do students deal with these dynamic terms together with the static definition of function?

Along the lines of the design of these activities, it could be interesting to design some new activities concerning other properties of functions in order to gain a deeper insight into possible exploits of the semiotic potential of functions' representation with parallel axes.

REFERENCES

Bartolini Bussi M. G., & Mariotti M. A., (2008). Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In L. English et al. (Eds.), *Handbook of International Research in Mathematics Education,* second edition, (pp. 746-783). New York and London: Routledge.

Carlson M., Jacobs S., Coe E., Larsen S., & Hsu E., (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352–378.

Carlson M., & Oehrtman M., (2005). Key aspects of knowing and learning the concept of function. Research Sampler 9. MAA Notes.

Confrey J., & Smith E., (1995). Splitting, covariation and their role in the development of exponential function. *Journal for Research in Mathematics Education, 26,* 66–86.

Dubinsky E., & Harel G., (1992). The nature of the process conception of function. In G. Harel and E. Dubinsky (Eds.), *The concept of function: aspects of epistemology and pedagogy*, (pp. 85–106). MAA Notes.

Falcade R., Laborde C., & Mariotti M.A., (2007). Approaching functions: Cabri tools as instruments of semiotic mediation. *Educational Studies in Mathematics, 66,* 317–333.

Goldenberg E. P., Lewis P., & O'Keefe J., (1992). Dynamic representation and the development of an understanding of functions. In G. Harel and E. Dubinsky (Eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy, 25*. MAA Notes.

Healy L., & Sinclair N., (2007). If this is your mathematics, what are your stories? *International Journal of Computers for Mathematics Learning*.

Ng O., (2016). Comparing calculus communication across static and dynamic environments using a multimodal approach. *Digital Experiences in Mathematics Education*.

Sinclair N., Healy, L., & Reis Sales C., (2009). Time for telling stories: Narrative thinking with Dynamic Geometry. *ZDM, 41*, 441–452.

Tall D., (1991) The psychology of advanced mathematical thinking, In: D. Tall (Eds.) *Advanced Mathematical thinking,* 3–21. Dordrecht: Kluwer.

Tall D., (1996). Function and Calculus, In: A.J. Bishop et al. (Eds.) *International Handbook of Mathematics Education,* (pp. 289–325). The Netherlands: Kluwer Academic Publishers.

Vinner S., & Dreyfus T., (1989). Images and Definitions for the Concept of Function. *Journal for Research in Mathematics Education, 20*(4), 356–366. National Council of Teachers of Mathematics.