STUDENTS’ COVARIATIONAL REASONING: A CASE STUDY USING FUNCTION STUDIUM SOFTWARE

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This paper presents a case study on students’ development of covariational reasoning, while using Function Studium software to perform activities about rate of change of linear and quadratic functions. This software was designed by LEMATEC-EDUMATEC/UFPE, a research group, and its development was guided by a model of software process based on Informatic-Didactic Engineering. The results of the case study pointed out some contribution of activities designed within the software to support students' covariational reasoning, such as: dynamic and simultaneous connections of the different representations of "rate of change" allowed the students to infer patterns of variation of these types of functions; and to coordinate average of rate of change to instantaneous rate of change.

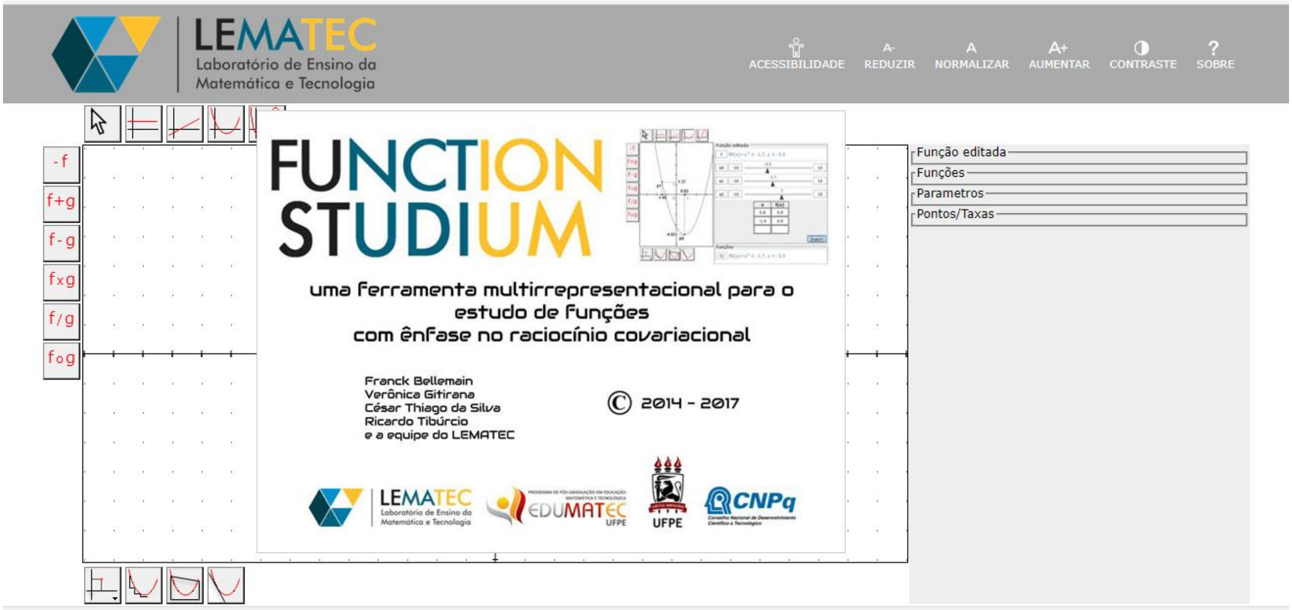
Keywords: function, covariational reasoning, educational software

INTRODUCTION

This paper presents a case study on students’ covariational reasoning using an *adhoc* software for the study of functions developed by LEMATEC-UFPE [1] (a Brazilian research group), as part of two master's dissertations (Tibúrcio, 2016; Silva, 2017).

The development of *Function Studium* software (Bellemain, Gitirana, Silva, & Tibúrcio, 2016) was guided by a model of software process that aims to combine aspects of teaching and learning mathematical concepts to computational aspects, contributing to a framework in both areas. The process model (Tibúrcio, 2016) is based on the idea of Didactic-Informatic Engineering (Bellemain, Ramos, & dos Santos, 2015), an object of study in LEMATEC - the research group.

Regarding *Function Studium* characteristics, the concept of rate of change and the variational perspective of functions underpin its tools. The ideas of covariational reasoning discussed in Carlson, Jacobs, Coe, Larsen, and Hsu (2002) contributed both to design the software tools and to build a framework for assessing their contribution on the development of students’ covariational reasoning after undertaken some activities within *Function Studium*.



FUNCTION STUDIUM

uma ferramenta multirrepresentacional para o estudo de funções com ênfase no raciocínio covariacional

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Figure 1. First screen of *Function Studium* Software

In this paper, we discuss some results of the case study, which was undertaken with a pair of preservice mathematics teachers of a Brazilian university. It is structured by: a brief discussion of the concept of covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002), the development of *Function Studium*, the methodology used in the case study and the analyse of some results obtained.

COVARIATIONAL REASONING

Covariation approach of function privileges the relationship between variables and how variation of one variable affects the variation of the other. Covariational reasoning is defined as "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002, p.354). The researchers designed a framework to analyse students' covariational reasoning while exploring dynamic situations of functions: five mental actions. According them, as students develop such reasoning, they advance in levels of covariational reasoning, as shown in Table 1

Mental action	**Description of the mental action**
MA1	*Coordinating the value* of one variable with changes in the other
MA2	*Coordinating the direction of change* of one variable with changes in the other variable
MA3	*Coordinating the amount of change* of one variable with changes in the other variable
MA4	*Coordinating the average rate of change* of a function with uniform increments of change in the input variable
MA5	*Coordinating the instantaneous rate of change* of the function with continuous changes in the independent variable for the entire domain of the function

Table 1. Mental Actions of Covariational Framework (Adapted from Carlson, Jacobs, Coe, Larsen, & Hsu, 2002, p.357)

The concept of rate of change assumes such an important role in the study of real functions under the variational approach. It is explored from elementary school to advanced courses for some areas, in which derivative is a central concept. Rate of change gains even more importance within the covariational perspective. Instead of emphasizing the correspondence among values, it focuses on how the variation of one variable affects the variation of the other, and so, it is expressed by the rate of change.

FUNCTIONS STUDIUM SOFTWARE DEVELOPMENT

To develop *Function Studium* software, a methodology that integrates principles from Didactic of Mathematics to Software Engineering process was built. This methodology is a proposal of implementing the educational software engineering principles (ESE) (Tchounikine, 2011) in the conception-development of a microworld for math teaching. More specifically, we worked in the integration of the firsts stages of Didactic Engineering (Artigue, 1996). The methodology takes into consideration methods of requirement, from software engineering, integrated with Didactic Engineering, which can be defined as an Engineering (Bellemain, Ramos, & dos Santos, 2015), in which we contemplate specifically theoretical potentialities (from teaching and learning the knowledge) and technologies (of computation).

The engineering phase (Bellemain, Ramos, & dos Santos, 2015) comprises essentially of four steps: delimitation of the field, theoretical, experimental, validation.

The step of delimitating the field aims to select the field of knowledge the software will exploit. In it, some questions were focused:

> […] which mathematics knowledge will be exploit within the software, what are the correlated knowledge that will also be needed to exploit, and son on and what kinds of professional can help in this development.
>
> (Tibúrcio, 2016, p.57, our translation).

The theoretical step comprised of a review of literature which aims to reach the state of the knowledge (didact, epistemological, cognitive and technological) regarding the selected field. It is the starting point of the process of requirements gathering. In the theoretical step, it is also important to address some questions which regards the didactic transposition (Balacheff, 1994): how computer potentialities will be used to digitally represents the knowledge domain; the way objects, relationships and operations are “internally” coded; and how they “dynamically” behave at the interface. In this step, it also starts software prototyping, in which: situations of use were designed, problems that could be raised while using the software were predicted; user’s answers were hypothesized; and the software prototype is developed to start some tests.

The experimental step comprises specific moments to test and to analyse the software prototype: interface, commands, bottoms, and so on, within the validation regarding of teaching and learning objectives traced within the situations of use.

The last step, validation, comprises an analyses a-posteriori of the results reached during the experimentation and the confrontation of this analysis with the theoretical one. This confrontation is made within the student's results for each activity, it gives us elements to improve the software as well as the situation of use. The step of validation can be done within different experiments, such as in a case study with pairs of students, and with the whole class. In this paper, we are still in the test with a pair of students.

Technologies used

Function Studium is a web-based software developed using HTML (HTML5), CSS and JAVASCRIPT. These languages, interpreted by any browser, can be edited with a simple text treatment and have already also innumerous object libraries what allows to shorten the period of development and to dedicate, almost exclusively, to implement the software codes regarding *the* didactic-informatic transposition of the concept of function and of rate of change, as well as the didactic proposal for these concepts.

The choice to use web platform also facilitates: to share it with the team involved in the Project, what helped to reconfigure the process of educational software engineering, mainly between the software engineering who conceive and technically develop it and the requirement engineering who worked in the conception/development regarding teaching and learning aspects. The use of an agile method and/or a methodology to facilitate a robust and quick interaction among the team was facilitated by this choice.

Function Studium Software

Function Studium presents a main window (Figure 2) rounded by icons which represent tools or configurations for the graphic representation, and others secondary Windows, in which others representations of function, such as tabular, algebraic, where implemented to interact with the graph.

Figure 2. Main Screen of *Function Studium*

Function Studium starts with one prototype of each type of function (constant, affine, quadratic and cubic), obtained by menu (2). It is based on the idea of an algebra of functions, which starts with some basic functions and operations defined (area 1). Other polynomial functions can be obtained as the result of an operation with these functions, as well as, the rational functions. In area 3, the window “Função editada” (Edited function) shows the algebraic representation of the function, while it is being inserted. In this window, it is possible to define the value of the coefficients of the function, using sliders. In area 4 (Functions), it shows the algebraic model of already defined functions. About area 5 (Parameters), it is exhibiting the coefficient of the already defined functions, what allows dynamic changes by sliders. Area 6 (Point/rates) shows the variable values, their variations and the rate of change of the functions in the select input values. It is possible to define such values both, directly through the window and through the graph. In area 7, there is tools to define a point in the graph. Regarding area 8, there is the tool "Rate of change", which, when activated, allows to calculate the rate of change of a selected function, both, between two points of the graph and between a sequence of successive intervals of x with the same length. Areas 9 and 10 refer to the tools "secant line" and "tangent line", in which it is possible to define tangent or secant lines to the graph of the function at the selected points, to articulate these lines with the rate of change of the function, contextualizing this concept with its geometrical meaning.

A detailed presentation and analyse which do not fit in this paper is necessary to clearly understand how the software works and to justify its conception, the chosen behaviour and articulations of the various representation registries of functions used in it. We suggest Silva (2016) and Tibúrcio (2016) to dive deeper in *Function Studium* elaboration. Concerning the functionalities to support and investigate the covariational reasoning of the students developed in the artefact, the conception followed the specifications elaborated during the requirements engineering process which basically specifies:

a) A covariational perspective: tools and characteristics that support students in the coordination of variable variation;

b) Dynamism, interactivity and different notations connected simultaneously: such characteristics are based on Kaput (1992), which synthetises possibilities of computational environments to instantiate variables and represent functions;

c) Tools and characteristics based on the results of the preliminary analysis on the concept of rate of change: these analyses pointed out elements of epistemology, teaching and learning of the concept, therefore, they should also support the software tools construction.

METHODOLOGY OF THE CASE STUDY

A Case Study was specially designed as a first validation of *Function Studium*. The study was undertaken with a pair of mathematics preservice teachers, who used the software to perform two activities about rate of change of linear and quadratic functions. The data were analysed to search contributions and/or limitations to the students' covariational reasoning derived from the use of the software.

The activities took place in a session of two and a half hours, in a classroom with the pair of students and the researcher, who had an observer role, except in moments in which it was necessary to interact with the students to solve doubts about the technical aspects of using the software. The students explored *Function Studium*, in a computer with internet access, and had to discuss to solve de problems proposed in a worksheet.

The activities focused on the last two levels of covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002), that is, on the coordination of the average rate of change and the instantaneous rate of change, although aspects of initial levels were naturally included in the questions. Some questions will be explicit in the analysis.

The data collected comprises a screen capture video with students’ interactions in the software, a video with students’ interactions between themselves, the notes on worksheets and a researcher diary. The data were analysed by the researcher, searching the students' answers and the dialogues associated with their actions on the computer screen, to identify extracts which explicit how the software interfered to the students' covariational reasoning.

ANALYSE OF THE RESULTS

As regard the last two levels of covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002), that is, on the coordination of average of rate of change and the instantaneous rate of change, the following results could be stated.

First, the dynamic variation of the independent variable in the graph, simultaneously connected to the dependent variable variation, allowed the students to coordinate the variation of the rate of change continuously (MA4). This allowed inferences about the rate of change behaviour in each type of functions addressed. An example is the students’ discussion about question 1.4 (Figure 3) when exploring a linear function.

Use the tool "rate of change" and in some interval of x, make the Δx decreasing in the graph or directly in the "points / rates" window (for example, 1; 0.5; 0.1; 0.05; 0.01; 0.001; 0.0001), observe

the behavior of the rate of change. In this software, when Δx is small enough, for example Δx = 0.0001, it is possible to simulate the limit of the function at point x_0 when x tends to x_0.
1.4) When simulating the process described above, take different linear functions and vary the variable x, observing the value of Δy / Δx. How does this value change as a function of x? What does this suggest about the rate of change in linear function?

Figure 3. Question 1.4

After the students varied x in the graph and observed, for a fixed Δx, and observed the behaviour of Δy, they inferred the behaviour of the related function.

Student 1: [...] "Delta x" is not changing correctly? Only x ... (He varies x in the graph and observes the invariance of the value of the rate of change for all the points reached). Do you understand? (He questioned whether his colleague had the same conclusion.). The value of "delta y" over "delta x" ... It doesn't vary. When you make delta x always smaller and then you change the value of x, it (Refers to Δy/Δx) will not vary.

Student 2: Ok…

Student 1: What does this suggest about variation in linear function? ... Humm ... Ah, that its variation is constant... What do you think?

Student 2: Thus, the rate of change of it is constant. It is always equal to the coefficient.

Second, the "rate of change" tool, which calculates the rate of change at successive intervals of the function domain and displays them both in graph and in points windows, was an important resource for students to coordinate the variation of average rate of change (MA3 and MA4) and to observe patterns of variation in quadratic functions. This can be seen in their discussion regarding question 2.3 while exploiting a quadratic function.

2.3) Still in the simulation of the previous item, vary the variable x in the graph and note the variation of the variation rate Δ (Δy/Δx) in the "Points/Rates" window. How does Δ (Δy/Δx) behave with the variation of x? Test other quadratic functions, describe what you perceive and what this suggests in relation to the variation in quadratic functions.

Figure 4. Question 2.3

Student 1 varies x in the graph and observes the "points window", in which the successive differences between the rates of change in the intervals of the graph are shown. Thus, he argued.

Student 1: In relation of the rate of change, when you change x, the rate of change will vary ... But, there, the difference between "delta y - two" over "delta x - two" and "delta y - one " over "delta x – one" is constant. The difference between the rates of change will be constant. That's interesting.

Third, their explorations of the "points window" helped them to coordinate the variation of average rate of change (MA4), since it exhibited in the same area the variation of Δx, Δy and Δy/Δx, simultaneously with the variation of x in the graph. Their discussion regarding question 2.1 shows that. Student 1 chooses the function $x^2–2x+2$ and uses the tool "rate of change" to change x in the graph while observes the variation of Δy in the points window.

Student 1: Look, I'm changing x here, and look at Δy ... Δx is the same, it's "one", and then Δy increases. Consequently, the rate of change, is that right? Because if Δx is always equal to "one" and Δy is increasing, then the rate of change will increase as well. Write there!

Student 2: How? What did you mean?

Student 1: Δy is increasing as you increase the values of x (...)

Fourth, as regard to the coordination of the instantaneous rate of change (MA5), as can be seen, students were able to vary the value of Δx in the "points window", making it closer to 0, while observing the variation of Δy approaching a specific value. In doing so, they could coordinate the transition from the average rate of change to the instantaneous rate of change, through smaller and smaller refinements of Δx, obtained in the "point window" and in the graph.

Fifth, the simultaneous connection of actions in different representations enabled within *Function Studium* supported them to coordinate the variation of the instantaneous rate of change, while varying x in the graph. This connection allowed them to explore aspects such as the signal and the variation of the rate of change as a function of x, as well as aspects of graph such as concavity and inflection points, from a variational perspective, relating these aspects to the behaviour of the rate of change.

CONCLUSIONS

This paper presented some of the results of a case study on covariational reasoning of a pair of students, who exploited *Function Studium* software within activities about rate of change of linear and quadratic functions. The results reveled that some characteristics and tools of the software supported the students' covariational reasoning till the last level (MA5), with emphasis on the simultaneous connection of different notations, which contributed to coordinate aspects such as the sign of the rate of change and patterns of variation of each type of function, as well as, it contributed to a variational interpretation of inflection points and concavity in the graph. Moreover, the dynamism within simultaneous variation of Δx, Δy and Δy/Δx allowed them to coordinate both the average and instantaneous rates of change, by means of smaller and smaller refinements in Δx.

NOTES

1. LEMATEC – Laboratory of Studies in Mathematics and Technology – of EDUMATEC - the Program of High Studies in Mathematics and Technologic Education – at UFPE - University Federal of Pernambuco - Brazil.

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