GEOGEBRA AUTOMATED REASONING TOOLS: A TUTORIAL WITH EXAMPLES

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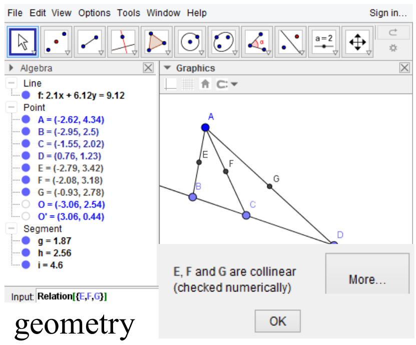
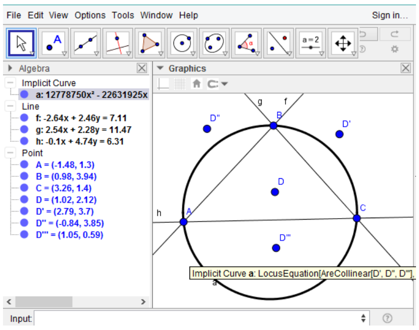
GeoGebra Automated Reasoning Tools (GGB-ART) are a collection of GeoGebra tools and commands ready to automatically derive, discover and/or prove geometric statements in a dynamic geometric construction. The aim of this workshop is to present, through examples, the use of GGB-ART and to argue about its potential impact in the classroom.

Keywords: Automated theorem proving and discovery, GeoGebra, Dynamic geometry software, Elementary geometry in education.

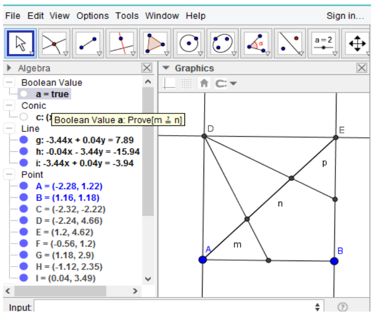
AUTOMATED REASONING IN ELEMENTARY GEOMETRY …



By "automated derivation of geometry statements" we refer to tools that, rigorously, output some/all geometric relations verified by a collection of selected elements within a geometric construction. For instance[1]: given a free point *A* and three points *B*, *C*, *D* on a line, consider *E*, *F*, *G*, the midpoints of segments *AB*, *AC* and *AD*. Then, the automatic derivation tool should output some property relating *E*, *F* and *G*.



By "automated discovery of geometry statements" we refer to algorithms that systematically find complementary, necessary, hypotheses for the truth of a conjectured geometric statement. For example[2], given a triangle *ABC* and a point *X*, let *M*, *N*, *P*, be the symmetric images of *X* with respect to the sides of the triangle. Then *M*, *N*, *P* are aligned. Obviously, this conjecture is false but…the automatic discovery algorithm should be able to output the necessary (and sufficient) location for *X* in order to have the alignment of *M*, *N*, *P*.



Finally, by "automated proving of geometry statements" we refer to algorithms that accept as input a geometric statement, such as[3]: "If two lines are drawn from one vertex of a square to the midpoints of the two non-adjacent sides, then they divide the diagonal into three equal segments". Then, the algorithm performs an exact computation (i.e. not using floating point numbers) and outputs a mathematically rigorous (e.g. not based upon a probabilistic proof) yes/no answer to the truth of the given statement.

The community of mathematicians and computer scientists has been working on these issues along the past 50 years, with a variety of approaches, outcomes and popularization results. See, for instance, the pioneer work of Gelertner (1959) in the A.I. context, or the algebraic geometry framework for automated reasoning in geometry, disseminated by the book of Chou (1988), that is behind our current implementation). Moreover, it is clear that the didactic perspective on proof (with or without technology) has been a research topic for over 40 years in the world of mathematics education (Richard, Oller & Meavilla, 2016).

… ITS GEOGEBRA IMPLEMENTATION

Hence, we consider quite relevant to present in this workshop a tutorial describing in detail the very recent implementation (2016) of tools and commands for the automatic deriving, discovery and proving of geometric theorems over the free dynamic geometry software GeoGebra, with a great impact in mathematics education. See: Abánades et al. (2016), Hohenwarter et al. (2016).

To begin with GGB-ART we have to draw in GeoGebra a geometric construction. Then we will exhibit the many possibilities that GeoGebra offers to enhance investigating and conjecturing about geometric properties of our construction. Say: investigating visually; using the Relation tool to compare objects and to obtain relations; or using the Locus tool to learn about the trace of a point subject to some constraints. These methods are usually well known by the GeoGebra community and well documented at the GeoGebra Materials web (https://www.geogebra.org/materials/). But these methods are mostly numerical, i.e. not mathematically rigorous, they only work on the specific construction with concrete coordinates, so they do not allow to deal with general statements.

GGB-ART brings to GeoGebra new capabilities for automatic reasoning in Euclidean plane geometry in an exact way, by using symbolic computations behind the concrete construction: the Relation tool and command can be now used to re-compute the results symbolically; the LocusEquation command refines the result of the Locus command by displaying the algebraic equation of the graphical output, allowing to investigate and conjecture statements; the Prove and ProveDetails commands decide in an exact way if a statement is true (i.e. checking the mathematical correctness of some previously found relation).

… ITS EDUCATIONAL IMPACT

Our final goal is to share these tools with the community of math teachers and math education researchers, aiming to improve, after suitably addressing the necessary changes and approaches in the educational context, geometry education (Botana, Recio & Vélez, 2017). This is an involved didactical issue, dealing with human reasoning with technology and with the validation modes available in the classroom (i.e. deductive, inductive and instrumental), so that the student can accomplish his/her mathematical work (Richard, Oller & Meavilla, 2016). It is not a new issue: in fact, it was already 30 years ago when educators started reflecting about the potential role in education of software programs dealing with automatic theorem proving (automatic discovery and derivation were inexistent at that time). See, for instance, the visionary ICMI Study "School Mathematics in the 1990's" (Howson and Wilson, 1986) or the inspiring paper by P. Davis (1995), with a section that refers to the "transfiguration" power of computer-based proofs of geometry statements. But these reflections were formulated rather as considerations about the future than as proposals for the present time of their authors…

Currently, although there already are some studies concerning the development of intelligent tutorial systems designed to assist students to construct proofs in geometry, such as GRAMY (Matsuda and Vanlehn, 2004), GeoGebraTutor or QED-Tutrix (Tessier-Baillargeon, Richard, Leduc and Gagnon, 2014) –as detailed in the most comprehensive review of existing tutorial systems, available in the comparative study of Tessier-Baillargeon, Leduc, Richard and Gagnon (2017)– it is fair to say that, up to now, the dissemination, use and impact of these achievements in the educational context is very limited. For example, another recent survey by Sinclair et al. (2016), on geometry in education, although it includes a full section on the role of technologies and another one on "Advances in the understanding of the teaching and learning of the proving process", does not refer at all about automated reasoning tools.

Thus, since the program over which we have implemented our automatic reasoning tools (ART) is currently available over computers, tablets and smartphones, with and without internet connection, i.e. on a well spread, dynamic geometry program, we think the time has arrived to consider the following question: what could be the role, in mathematics instruction, of the ample availability of such tools? In this direction, our final goal is to make an open call to the community of math teachers and math education researchers, to join us preparing a research project to address the following issues: Are ART in geometry education good for anything? If yes, what are they good for? What should be the necessary changes and requirements in the educational context, if ART are to be considered good for anything?

…AND DIDACTIC FRAMEWORK

It is easy to consider the ART as an authentic geometric calculator. First, because they determine equations, even measures, and above all because they link different effects to help discovering new properties or to produce valid reasoning, like propositional calculus. We can consider the benefits or drawbacks of geometric calculators from a user perspective, here the teacher or the student who exploits them in school. In the same way that conventional or graphical calculators do not reveal the models on which the algorithms are based, the ordinary user of geometric calculators does not have access to the models that run them and produce answers. However, from a behavioral perspective, GeoGebra ART is not merely a black box that produces effects or reactions to actions determined by a waiting user. In fact, just as the ancients were questioning an oracle to predict what would happen in a given context, the user employs an ART as a guiding stick in the geometric environment.

Indeed, with regard to the theory of didactical situations in mathematics of Brousseau[4], we can see the ART as belonging to the *milieu*, that is to say, as being a playing partner of the student in the construction of knowledge. Of course, the *milieu* conveys knowledge and it is the model implemented in the tool that determines the need for it. However, the need for the student in interaction with the *milieu* can be quite diverse. In the case of ART we regard this interaction as follows: the student works in a situation (context, problem or task), questions the *milieu* in the particular logic of the situation and in a more general logic of the didactic contract that binds him or her to the knowledge at stake. He or she wants answers to fit the context, to solve the problem or to accomplish the task; he or she probably does not need to mobilize all logical artillery of mathematical proofs with its particular mode of expression and its high epistemic value. In terms of reasoning, ART helps producing genuine abductions, in the sense of Pierce, which facilitates the student inquiry into the situation, even when he or she was trying to solve a problem of geometric formal proof.

Several works have already dealt with the merging of mathematical proofs, visualization and dynamic geometry, but surprisingly, references to other natural links with geometry are often missing in the literature. If we consider the work carried out in the working groups on geometrical thinking, as in the CERME (for details, see Kuzniak, Richard & Michael-Chrysanthou, 2017), we can mention that few research works focus on modelling of physical phenomena using geometrical tools, or deal with solving problems in geometry that are not problems of proof, or go beyond the mere discovery of some characteristic properties well defined and known in advance by the teacher and by the student. However, the very constitution of the geometric model by the student is certainly an incarnation of what modelling of form, shape and space is. Unfortunately, modelling activity is generally not widely practiced in compulsory education, and problem solving in geometry classes is often limited to those based on well-defined tasks. Moreover, few studies concern the solving of open problems or those that require a problematization which is not already linked to a geometric model known in advance. In this context, we believe that the functionalities of ART are particularly useful in supporting the development of mathematical competencies through the development of a geometrical culture, building on mathematical discovery and modelling approaches.

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[4] See, for some of his key works, translated to English, http://www.springer.com/gp/book/9780792345268