REASONING STRATEGIES FOR CONJECTURE ELABORATION IN DGE

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The present study analyzes students' reasoning strategies for elaboration of conjectures when working in a Dynamic Geometry Environment (DGE). We observed 18 pairs of ten-graders in a private school in Lebanon, while working on open geometrical proof problems using Dynamic Geometry Software (DGS), namely GeoGebra. The analysis revealed three reasoning strategies employed by students. In the first two, the students worked on satisfying the presumed premise of the conjecture in the figure and identifying / validating the conclusion, either by observing the figure at hand (strategy 1) or by dragging the figure to validate the conclusion across different instances (strategy 2). Conversely, in the third strategy, the students worked on satisfying the conclusion of the conjecture in the figure and observing it to identify the premise that corresponds. Each strategy entails the use of different construction tools and types of constructions which affect the correctness of the resulting conjecture.

Keywords: secondary – conjecture – dynamic geometry – reasoning

SETTING THE CONTEXT

The proving process involves two sub-processes: conjecture elaboration and proof development. These two processes become particularly more explicit in Dynamic Geometry Environments (DGEs) since the nature of the first process in DGE is radically different from pencil-and-paper environments, which consequently affects the way the second process evolves. Dynamic draggable constructions strongly affect the proving process by mediating the type of conjectures developed (Sinclair & Robutti, 2012).

Extensive research (Hölzl, 2001; Laborde & Sträßer, 2010; Laborde & Laborde, 2011; Laborde, 2005) has been conducted on the potentialities of the dragging tool, which resulted in understanding this tool as a pedagogical tool conducive to mathematical reasoning (Jones, 1998), particularly in the process of conjecture formation in geometry. The epistemic potential of the dragging tool lies in its relationship with the discernment of invariants (Leung, Baccaglini-Frank, & Mariotti, 2013). According to Mariotti (2014), dragging acts as a mediator between geometrical invariants and logical statements. In fact, dragging to elaborate a conjecture is a complex process as it requires the interpretation of perceptual data by analyzing the image in order to identify a geometrically significant relationship between its elements and properties. For example, when dragging to search for consequences, students need to interpret the geometrical dependence between direct invariants (i.e. invariant properties observed between independent elements) and indirect invariants (i.e. invariant properties observed between dependent elements) as the logical dependence between the premise and the conclusion of a conditional statement.

Given that in the literature the primary focus in the process of conjecture elaboration has been on the role of dragging, this study aims at a more comprehensive analysis of the process of conjecture elaboration within DGE by shifting the focus from the dragging tool to include the different reasoning strategies, construction tools and types of constructions employed by students. The study consisted of a series of observations of 18 pairs of ten-graders (15-17 years old) who worked on open geometrical proof problems within a DGE, namely Geogebra. Data were collected using video-recording and collection of materials, including any paper trace (sketches, scribblings, proof formulations) generated by students, together with their GeoGebra files. This paper presents students' work on two problems previously used by Olivero (2002) and Arzarello et. al. (2002) respectively:

Problem 1 (P1) (Olivero, 2002)

1) Let ABCD be a quadrilateral. Consider the bisectors of its internal angles and the intersection points H, K, L, and M of pairs of consecutive bisectors.
2) Drag ABCD, considering different configurations, and explore how HKLM changes in relation to ABCD.
3) Write down conjectures and prove them.

Problem 2 (P2) (Arzarello et. al., 2002)

Given a triangle ABC, consider P the midpoint of [AB] and the two triangles APC and PCB. 1) Explore the properties of the triangle ABC which are necessary so that both APC and PCB are isosceles (in this case, the triangle ABC is called "separable").

2) Write down conjectures and prove them.

STRATEGIES FOR CONJECTURE ELABORATION

Through observing and analyzing the students' work, it was possible to identify three reasoning strategies that were used for conjecture generation and that describe the way students moved from exploring the changes of the dynamic geometrical figure to identifying relational properties of the figure. A sentence is considered to be a conjecture when stated in the form "If… then…." the first part (if…) is referred to as premise and the second part (then…) as conclusion. Upon analyzing students' work, the following strategies for conjecturing were identified.

Strategy 1 – Forward Static Observation (FSO)

This strategy consisted in building a figure and satisfying the presumed premise, then observing a static instance of the figure to conclude the conjecture. Students construct the figure while incorporating the properties provided in the presumed premise, then observe, without any manipulation, a single static instance of the figure obtained, to identify invariants and to formulate a conjecture. The constructed figure may be robust or visually adjusted.

The inferences made while using the FSO strategy are of two types:

- *Type 1 inference – visual verification*: When the conclusion was a priori known by the students, i.e. provided by the statement of the problem (as in problem 2 where students knew they had to find a separable triangle ABC), students incorporated the presumed premise and observed whether the single instance of the geometric figure obtained met the conclusion. In such a case, a conjecture was developed; if not, then the premise was rejected based on the counterexample.
- *Type 2 inference – visual speculation*: When the conclusion was not provided within the problem (as in problem 1 where students did not know which shape of HKLM they will obtain), then, after incorporating the presumed premise, students identified the conclusion based on a single instance of the geometric figure and formulated a conjecture.

The following examples illustrate different cases of use of the FSO strategy for conjecturing. The code between parentheses refers to the problem being solved (P1 or P2).

Example 1 (P1). Many students who were working on problem 1 started by investigating the case - premise: "If ABCD is a square". They constructed a robust square using the *Regular Polygon* tool (Figure 1). Not knowing what the conclusion should be (type 2 inference), they observed the figure and developed the conjecture: "If ABCD is a square then H, K, L and M coincide".

Even though the conjecture was based on the mere observation of a single figure, the fact that it was a robust construction provided greater validity and reliability for the conjecture.

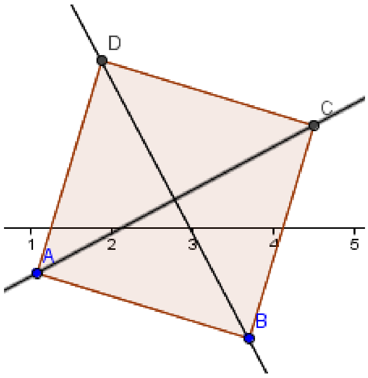

Figure 1. Constructing a robust square and observing a static drawing to conclude

Example 2 (P1). Another pair of students constructed a parallelogram using the Parallel Line tool, then the angle bisectors of the angles and dragged A, B and D to form a square by visual adjustment (Figure 2). Similarly to example 1, they did not know what the conclusion should be (type 2 inference), so they observed the figure and developed the following conjecture: “If ABCD is a square then HKLM is also a square”.

However, in contrast to example 1, their conjecture was based on a single instance of a soft figure obtained by visual adjustment, in which case the drawing was inaccurate, yielded a rectangle instead of a square, and led to an incorrect conjecture.

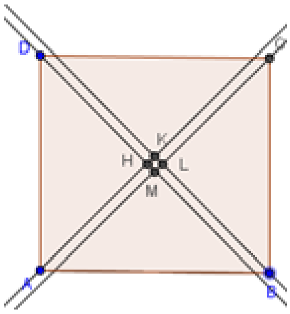

Figure 2. Constructing a square and observing a static drawing to conclude

Example 3 (P1). One of the observed pairs of students dragged the vertices of the scalene polygon ABCD to make it a trapezoid, based only on visual adjustment (Figure 3). As they did not know what conclusion to expect (type 2 inference), they observed the figure and formulated the conjecture: “If ABCD is a trapezoid then H, K, L and M coincide”.

The students observed in a single figure that H, K, L and M coincided and generalized the result to the entire class of trapezoids ABCD without validating the conjecture in additional instances of the figure. Thus they were not able to observe the different types of quadrilaterals HKLM obtained for different types of trapezoid ABCD.

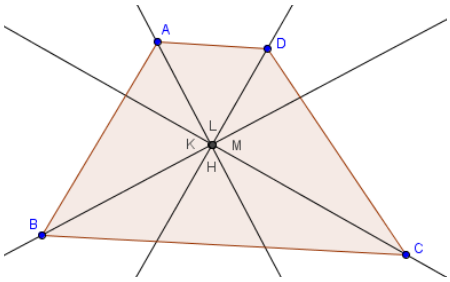

Figure 3. Constructing a trapezoid and observing a static drawing to conclude

Example 4 (P2). A pair of students investigated the premise, “ABC equilateral triangle”, by drawing a scalene triangle ABC, constructing the perpendicular bisector of [AC], dragging B onto it to make the triangle isosceles, displaying the measure of $\widehat{ABC}$, and dragging B along the perpendicular bisector until having $\widehat{ABC}$ = 60º (Figure 4). Since the students were given the conclusion they are supposed to reach, i.e. ABC separable (type 1 inference) they rejected the premise since in that figure ABC was not separable.

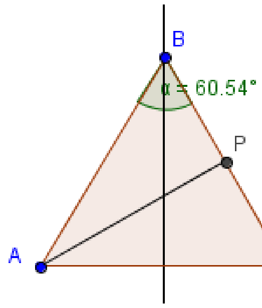

Figure 4. Constructing an equilateral triangle and observing a static drawing to conclude

Strategy 2 – Forward Dynamic Observation (FDO)

This second strategy consisted in constructing a robust figure satisfying the presumed premise, then dragging to search for the invariant properties and conclude the conjecture. If the invariants were observed across dragging, the conclusion is identified and a conjecture is developed (examples 5 and 6); if not then the premise is rejected based on a multitude of counterexamples (examples 7 and 8). To use this strategy, the construction is required to be robust in order to hold under dragging.

Example 5 (P2). A pair of students wanted to explore if the triangle ABC is separable when it is right at C. They built a robust figure that satisfied their premise (i.e. ABC right) using perpendicular lines (CA) and (CB) (Figure 5). They dragged the independent points to identify invariant properties across dragging (i.e. two equal sides for each of the triangles CPA and CPB) which was their conclusion.

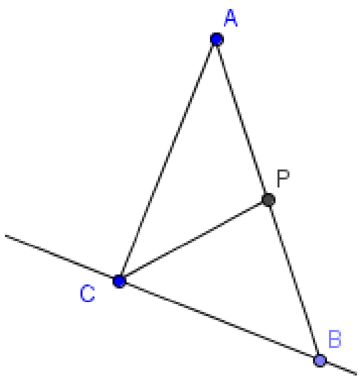

Figure 5. Constructing a robust right triangle and dragging to identify invariants

Example 6 (P1). One pair of students chose to investigate the premise “If ABCD is a parallelogram” to determine its conclusion. They constructed a robust parallelogram using parallel lines (Figure 6). They dragged A, B and D and identified the invariant properties of HKLM and deduced that it is a rectangle. They formulated the conjecture: “If ABCD is a parallelogram then HKLM is a rectangle”.

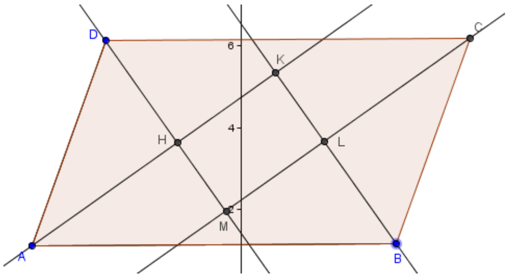

Figure 6. Constructing a robust parallelogram and dragging to identify invariants

Example 7 (P1). One of the observed pairs of students, attempted to investigate the premise: “If ABCD is a trapezoid”. They constructed a robust trapezoid using the *Parallel Line* tool. They dragged the vertices of ABCD (Figure 7) but were not able to identify any invariant property for HKLM through dragging. Thus the case of the trapezoid was rejected.

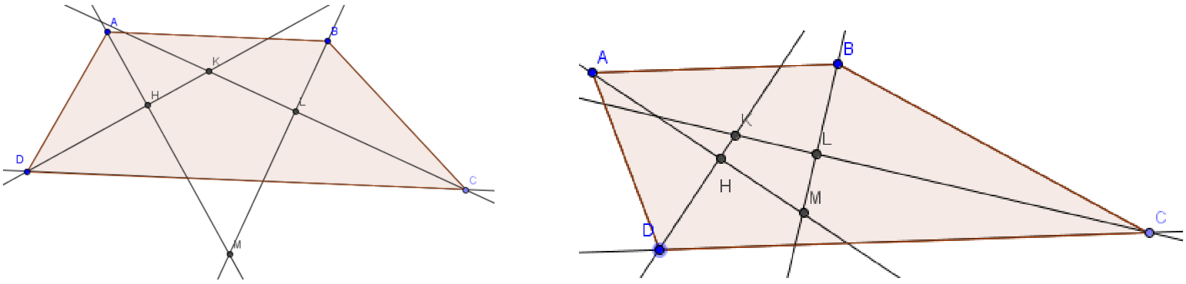

Figure 7. Constructing a robust trapezoid and dragging to identify invariants

Example 8 (P2). A pair of students wanted to explore if the triangle ABC is separable when it is isosceles. So they built a robust figure that satisfied their premise (i.e. ABC isosceles) by placing a point F on the perpendicular bisector of a segment [DE] and formed a robust isosceles triangle (Figure 8). They dragged the independent point D to identify invariants of the conclusion i.e. two equal sides for each of the triangles DGF and GFE. Given that the conclusion could not be met, the premise was rejected.

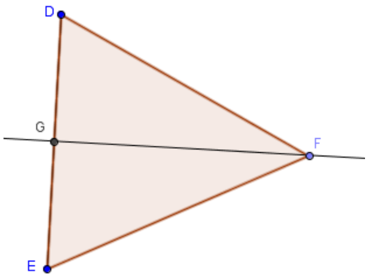

Figure 8. Constructing a robust isosceles triangle and dragging to identify invariants

Strategy 3 – Backward Static Observation (BSO)

This third strategy consisted of incorporating the conclusion of the desired conjecture in the figure and observing that single static instance of the figure to discover the corresponding premise. Students’ assumption is that, if they are able to incorporate the properties of the conclusion into the construction, the drawing should reveal the premise for which they are looking. If it is not possible to incorporate the properties of the conclusion in the construction, the premise should be rejected.

In problem 2, although the conclusion is given by the problem, it can be further developed into sub-conclusions; CPA and CPB can be considered simultaneously isosceles at different vertices. Students have actually attempted testing different sub-conclusions.

Example 9 (P2). A pair of students satisfied the conclusion of the conjecture in their figure by making ABC separable (APC and PCB isosceles at P and C respectively). To do so, they sketched a scalene triangle ABC, constructed the perpendicular bisectors of [AC] and [BP], and then dragged the vertices of ABC to bring P and C simultaneously onto the respective perpendicular bisectors (Figure 9). They were able to observe in the figure that the conclusion (i.e. APC and PCB isosceles at P and C respectively) is satisfied when ABC is a right triangle, thus completing their conjecture. We note that, in this case, PCB is equilateral, but students were not aware of this fact.

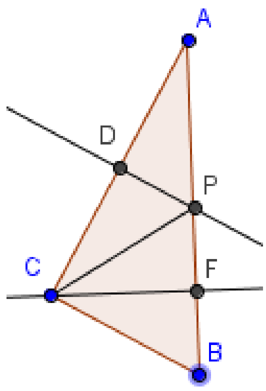

Figure 9. Satisfying the conclusion "ABC separable" and identifying the premise "ABC right triangle"

Example 10 (P1). After developing the conjectures "If ABCD is a square then H, K, L, and M coincide" and "If ABCD is a rhombus then H, K, L, and M coincide", one pair of students attempted to investigate under which condition (in general) do the points H, K, L and M coincide. They dragged the vertices of ABCD to form a new quadrilateral ABCD where H, K, L and M coincided; that is they satisfied the conclusion "H, K, L and M coincide" in the figure. They formed a kite shape (Figure 10). However, they were not able to identify its nature and thus were not able to develop a general conjecture on the nature of ABCD for which H, K, L and M coincide.

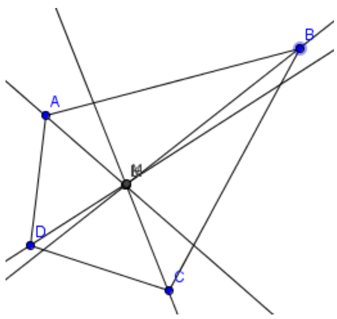

Figure 10. Satisfying the conclusion "H, K, L and M coincide"

Example 11 (P2). One pair of students thought about investigating whether there is a specific premise, i.e. nature of ABC, which satisfies the conclusion "ABC separable at C", that is CA = CP and CP = CB. They constructed AC = 5; CB = 5 and connected A and B. Then they constructed CP = 5 and tried to drag P onto [AB] but were not able to do so (Figure 11). Thus they concluded that it is an impossible case.

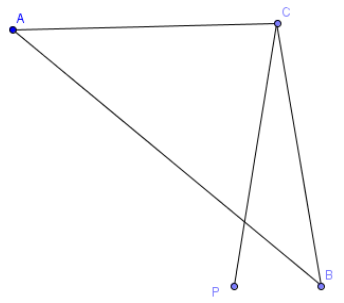

Figure 11. Satisfying the conclusion "ABC separable at C" to identify the premise

CONCLUSION

The analysis of students' work provided in the study allowed us to identify three strategies (Figure 12) for the elaboration of conjectures based on the way students move from exploring the different instances of the figure to identifying relational properties of the figure and thus developing conjectures. In the first two strategies the students work on satisfying the premise in the figure and identifying the conclusion by observing the static figure at hand (strategy 1 - FSO) or by dragging a robust figure to validate the conclusion across different instances (strategy 2 - FDO). Conversely, students can also work on satisfying the conclusion in a soft figure and observing it, without any manipulations, to deduce the premise that corresponds (strategy 3 - BSO).

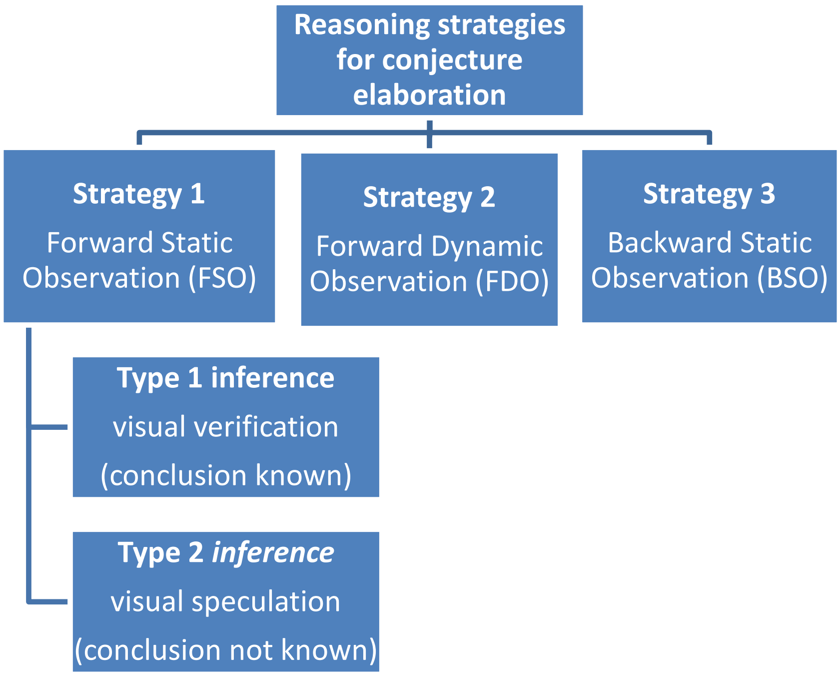

Figure 12. Reasoning strategies for conjecture elaboration

The weakness of FSO is caused by the elaboration of the conjecture based on a single instance of each case; students are only observing the figure they constructed and formulating the conjecture based on that single instance of the figure which, in many cases, happened to have additional properties leading to a misguided conclusion.

In FDO, the use of robust constructions and dragging tool lead to the creation of a powerful instrument for conjecture generation since the use of robust constructions results in valid drawings, which eliminates ambiguous results. Also, when dragging robust constructions, the premise can be verified in a multitude of figures. Thus the elaboration of a conjecture or the rejection of the premise is based on a multitude of instances of the same case.

In contrast to the first and second strategy, in BSO the students work their way backwards from conclusion to premise, which is not always an easy task. Most students preferred strategies 1 and 2, that is testing different premises to find the one that satisfied the conclusion instead of incorporating the conclusion into the construction and letting the figure reveal the premise i.e. strategy 3. However, based on a single instance of the figure, students may consider a certain observed property to be the premise, when in fact it is not. The premise has to be induced from invariants across dragging.

The identification of these three strategies induces thinking about a fourth possible strategy that was not observed in the participating students' work but that we can see as a possible one. When working from premise to conclusion, students used both soft (FSO) and robust (strategy 2) constructions. However, when working from conclusion to premise, only soft constructions were used (strategy 3). Therefore, we develop this potential fourth strategy in order to hypothetically describe the work from conclusion to premise using a robust construction. The strategy would be named "Backward Dynamic Observation (BDO)" and would consist in satisfying the conclusion by construction and dragging to deduce the premise that led to the desired conclusion. The use of robust constructions is required to ensure the validity of the figure and allow the elaboration of the conjecture based on a multitude of instances of the same case through the use of dragging. More research with a larger sample of students is needed to validate the fourth potential strategy.

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