ALGEBRA STRUCTURE SENSE IN A WEB ENVIRONMENT: DESIGN AND TESTING OF THE *EXPRESSION MACHINE*

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The article reports on outcomes from a study that aims to investigate the role of affordances, level-up and feedback in the web environment Expression Machine in developing the algebra structure sense of tertiary education students. Algebraic substitution is the main procedure involved in the way the software works. Its design and testing methodology are based on the Human - Computer Interaction aspect of Activity Theory. From this approach, the study redefines the notion of algebra structure sense formulated in previous works. Results from the experimental sessions show that some features of the environment favor the development of sutudents structure sense, specifically when they deal with substitution and factorization tasks. At the same time, it was possible to identify aspects to be improved, for instance, adding categories of tasks with increasing structural complexity and less visually salient, which may require a greater cognitive demand from students.

Keywords: Algebra structure sense, web environment, human-computer interaction, activity theory.

INTRODUCTION

Once they have overcome the difficulties of learning the rules of syntax, as well as those for understanding the semantics of symbols and the conventions of algebraic notation, students in tertiary education face the challenge of recognizing the basic structures of algebraic expressions in complex transformational algebra tasks. Hoch & Dreyfus call this recognition ability ‘algebra structure sense’ and they define it as a set of abilities that involve: 1) recognizing a familiar structure in its simplest form; 2) dealing with a compound term as an entity and, by performing the appropriate substitutions, recognizing a familiar structure within a more complex form; and 3) choosing appropriate manipulations for a better use of structure (Hoch & Dreyfus, 2007, pg. 436).

As of the above definition, these authors designed activities to be used during teaching interviews with 11th grade students. A pre-post test scheme and an analysis of the interview protocols revealed that the students made progress regarding the development of structure sense in specific cases, such as applying the rule $a^2 – b^2 = (a + b) (a - b)$ in compound expressions such as $(x + 8)^2 – (x - 7)^2$ or $x^2 – (x + 1)^4$. However, the results also showed that students found factoring variants like $(x + 3)^4 – (x – 3)^4$ extremely challenging, despite their having the support of the researcher. This shows that the research area is emerging, especially when investigating how to teach structure sense. Formulation of the project ‘Developing structure sense with digital applications’ was largely inspired by the work of Hock and Dreyfus. This project intends to deepen the research of the learning and teaching of the structural aspects of symbolic algebra within technology environments. The study reported here was undertaken in the framework of this project and its main goal is to investigate the role of *affordances, level-up* and *feedback* [1] in the virtual environment *Expression Machine* in developing the algebra structure sense of tertiary education students. This article briefly describes the design features of the *Expression Machine* and reports its testing results, specifically with tasks that involve algebraic substitution and factorization [2].

BACKGROUND

Structure Sense

Prior to the work of Hock & Dreyfus, the topic of structure sense in algebra was studied by A. Arcavi, who tried to characterize symbol sense by extrapolating part of the information from number sense. The latter is conceived as 'a non-algorithmic sense of numbers', based primarily on an understanding of its nature and the nature of its operations, as well as the need to examine the good sense of its results and related effects. The author establishes parallels with this conception by referring to symbol sense as the complementary relationship between algebraic manipulation and 'seeing through' algebraic expressions (i.e *seeing the unseen*, Arcavi, 1994).

On the one hand, in 2004 D. Kirschner found that students spontaneously respond to the visual patterns of algebraic expressions (visual salience) independently of the declarative rules, which suggests that typical errors like a + x / b + x = a/b reflect the predominance of visual aspects over the declarative knowledge of algebraic rules. According to this author, the receptive disposition of students to the visual structure of rules, independently of their intellectual commitment to the declarative content, is at odds with the habitual cognitive presumption that human intellectual abilities rely on the acquisition or development of algorithms and well structured rules (Kirshner, 2004, pg. 4). From this, Kirshner concludes that absent an understanding of the structural fundamentals, what students register is something about the visual shape of correct and incorrect applications, and that eventually, with persistence, the visual pattern recognition processes become sufficiently refined that they may restrict incorrect applications (2004, pg.42).

On the other hand, Sfard and Linchevski (1994) have documented the persistence of students to remain within the procedural aspects of algebra. That is, students tend to interpret algebraic expressions as calculation processes, and after repeatedly applying a procedure (or algorithm), they see them as objects, something upon which to reflect. They call this phenomenon reification. From this perspective, according to these authors, algebraic expressions have a dual process/object nature.

In very different ways, the studies of Hock & Dreyfus, Arcavi, Sfard & Linchevski, and Kirshner state that independently of what is meant by the nature of structure sense, implementing it in symbol manipulation tasks is enormously complex. Furthermore, the conclusions reached by these authors suggest that teaching structure sense is extremely challenging. The research shown in this paper intends to face this challenge by using the potential of technological resources for the learning of mathematics.

Algebra Learning with Technology

There is currently a broad repertoire of technological tools that can be used to teach algebra at various school levels, most notably; Computer Algebra Systems (CAS), Spreadsheets, Aplusix and the widely used Geogebra, which combines several mathematical representations (graphical, algebraic and geometric). In most cases, learning activities are focused on topics of functions, equations and graphs, as well as the use of CAS to verify the results of solving equations or performing algebraic expression transformations by hand (for instance, simplifying or developing expressions). The literature reporting results of research undertaken using these tools for teaching and learning algebra is significant and provides evidence of their great didactic potential. However, the literature on the use of software designed for teaching specific topics is less abundant. Some examples of these types of environments are: *eXpresser*, especially designed to foster generalization processes in algebra (https://migenproject.wordpress.com/using-migen/); *Virtual Balance,* used to teach the solving of linear equations (Rojano & Martínez, 2009); and the program *DragonBox* (www.dragonbox.com) which has the features of serious games [3], developed around entertainment in solving algebraic tasks and integrating these activities in the game.

STRUCTURE SENSE AND THE *EXPRESSION MACHINE*

To undertake the study reported here, the web environment *Expression Machine* (EM) was designed. EM is an ad-hoc tool for developing structure sense among tertiary education students, inspired by *serious games* and *touch* applications, with virtually no instrumentation time (training time at the use of the artifact level). The guiding resarch questions are:

1. Is it possible to guide students, through *affordances* and *feedback*, towards actions that allow them to perform tasks fostering development of an algebra structure sense?
2. Specifically, what features of *affordances* and *feedback* in a virtual environment foster students developing a structure sense for algebraic substitution and factoring expressions?

Theoretical Elements

Regarding the notion of structure sense underlying the design of EM, the principle of algebraic substitution is the main consideration, which allows for equivalent symbols to be used interchangeably, so one may be used instead of the other in an algebraic expression; a variable may be replaced by an expression and vice versa (Freudenthal, 1983, pg.483). In terms of the way that a structure sense may be acquired or developed, we resort to the idea, on the one hand, that meanings arise during usage and activity in practices that are shared socially within a community (the second Wittgenstein, 1988); and on the other hand, the idea that meanings are associated with training, following rules and *seeing how* (Huemer 2006).

EM software design and testing methodology are based on the Human - Computer Interaction (HCI) aspect of Activity Theory, with special emphasis on the notion of *affordances* or preconditions for action.

In the Activity Theory (AT), activity in general -not just human activity- but rather the activity of any subject, is understood as an intentional interaction of the subject with the world –a process in which mutual transformations take place between the “subject-object” poles. In this theory, the subject-object relationship is a starting point and it is interpreted as a non-direct relationship, that is to say, that it is mediated by language and artifacts, and as a non-symmetrical relationship because in it the subject holds the initiative and command.

In the field of HCI and of designing digital artifacts, the foregoing is translated into having the relationship between two components of a large scale system be asymmetrical, given that the interaction is begun and undertaken by the subject so as to cover its needs (Kaptelinin & Nardi, 2006, pg. 30). As such, an activity consists of a person or several persons doing something toward attainment of some end. In the field of learning, according to Knutti (1996), an activity is a way of doing that is oriented towards an objective, and learning is strongly linked to the doing and the social system in which the doing takes place. From this perspective, technologies are not a means by which knowledge is transmitted to a user, rather a tool that provides structure and mediates learning through activity (DeVane & Squire, 2012, pg. 242).

The AT envisages learning technologies not as ‘teaching machines’, but rather as ‘a support system for learning by doing’. Learning is not only accomplished through observation, but also by ‘doing’, and learning technologies serve to support and structure those tasks (Knutti, 1996, pg. 26).

Taking this approach, the study redefines algebra structure sense in terms of actions, as follows: A student demonstrates having algebra structure sense if, in order to solve an algebraic manipulation task efficiently, the student performs a combination of the following actions: a) Recognition of structures (for instance, recognizes notable products); b) See-how, that is, switching between various forms of an expression, to take advantage of the structures (learn to see sub-expressions as an object or entity); c) Substitution (whether internal or explicit); and d) Timely application of known algebraic identities.

Characteristics of *Expression Machine*

EM is a web application that was developed for users to learn, through experimentation and practice, the rules that the machine uses to generate tasks and, in time, acquire an algebra structure sense, in terms of actions a) to d). It incorporates school algebra rules such as algebraic substitution and equivalence of expressions. The interactive sequence was designed as of a scheme where the elements of the machine are the *input* (two expressions IE1 and IE2), *process* (a generating expression GE) and *output* (a resulting expression OE after substituting IE1 and IE2 in GE) (see Figure 1a).

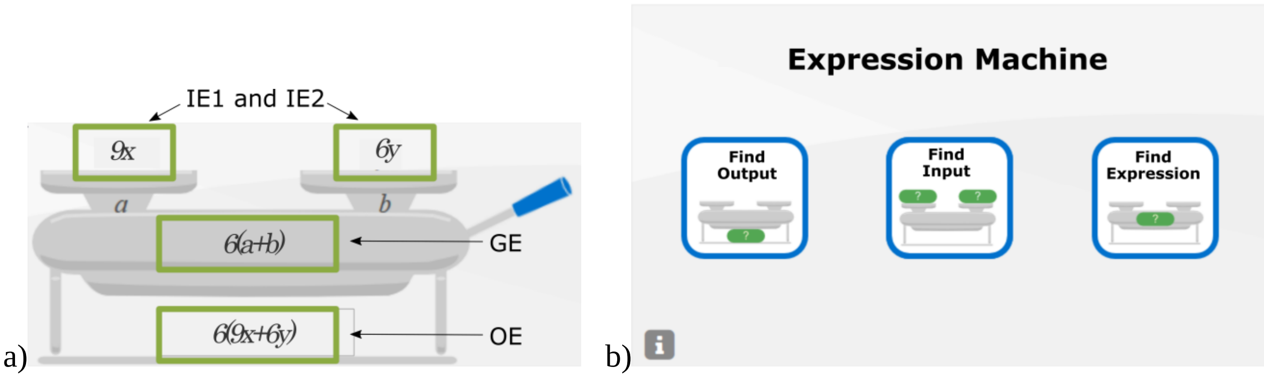

Figure 1. a) Expression Machine b) Main screen

In all cases, the expression machine generates an output expression upon substituting the input expressions in the generating expression. For instance, if the input expressions are *9x* and *6y*, and if the generating expression is 6(a+b), then the machine will output *6(9x+6y)* as it substitutes *a* with *9x* and *b* with *6y*. The activities proposed with this machine are of three types, which can be accessed through the main screen (see Figure 1b). These activities are:

1. *Conjecture input expressions*. Given an output and a generating expression, students must describe two input expressions that would produce such output expression.

2. *Predict the output expression*. Given the input expressions and a generating expression, students must describe the output expression the machine would produce. Students are asked to write down the expression they believe the machine will produce. They may then process the expressions and get the machine to produce an output expression. With this activity, in addition to practicing algebraic substitution, students may strengthen their knowledge on equivalent expressions, as the machine may give an equivalent expression that is syntactically different from their prediction. Feedback from the machine will clarify that the expression they input is correct and equivalent to that given by the machine.

3. *Conjecture generating expressions*. Given input and output expressions, students must conjecture a generating expression that will make the machine produce the given output expression. They may key in the generating expression into the machine to prove that it really works.

In most of the cases, interaction with EM requires intensive algebraic manipulations to be performed by hand (Muñoz & Rojano, 2014).

EXPERIMENTAL WORK WITH EM

The EM was tested with a group of 35 tertiary level students in a Mexico City public school. A pre-post test scheme (based on the questionnaires of Hoch and Dreyfus) was applied to assess participants' mastery of symbolic manipulation and their structure sense level. Participants had a period of interaction with the EM between tests. 16/35 of the students correctly solved 15 or more of the 32 items in the pre-test, and seven of the students took part in the experimental in-person sessions (with the participation of the researcher to briefly explain operation of the EM). In that 1.5-hour session, the students worked intensively on the three types of EM activities ('find the output', 'find the input', and 'find the generating expression', see Figure 1b) and they worked through different levels of complexity in terms of the algebraic expressions involved. 'The doing' of the students included paper and pencil algebraic manipulations to solve the three types of EM activities. Their actions were recorded during this experimental period, and the method described in Bødker (1995) was used to analyze recordings that detected *focus shifts* and *breakdowns* [4].

After the in-person experimental session, they were given the URL of the EM web application for them to use it freely at home over the course of one week. The post-test was applied at the end of that week of home use. In summary, two types of data were collected, as follows: 1) data collected with application of the pre and post-tests, where the written algebraic productions of students were analyzed; and 2) the material entailed in the video-tape of the interactive experimental (in person) session with the EM, together with the respective paper and pencil productions of the students, which show the algebra manipulations that they undertook in order to solve the EM exercises (see Figure 2).

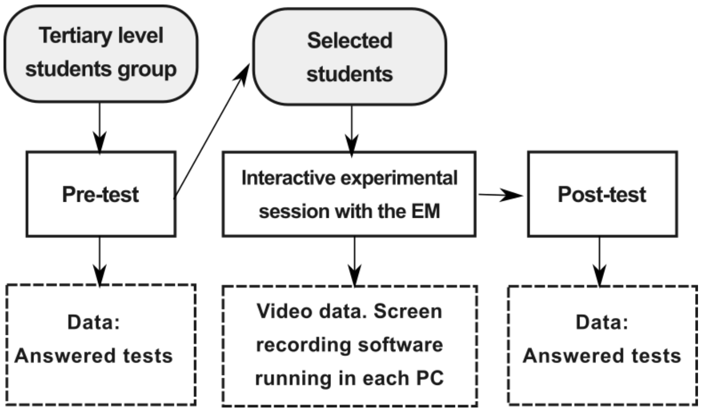

Figure 2. Types of collected data

Pre – Post Test Results

As a group, students improved significantly between the pre and the post test in both items of algebraic manipulation, and in items related to structure sense (according to the re-definition of structure sense formulated in this study in terms of actions a)-d)). One performance of particular merit was noted (Bedani), as it showed algebraic skills that surpassed those of the rest of the group. In order to illustrate this progress, two extracts taken from the productions of Edwin in the pre and post tests are presented.

In problem 9 (Figure 3) Edwin shows a good level of algebraic manipulation in the pre-test but fails to recognize *5-x* as an entity. However, he correctly factorizes *7-y* in the post-test in order to obtain a product of two binomials, and quickly solves the problem. In the latter case, actions a) and b) become evident. Similarly, in problem 12 of the pre-test, Edwin applies several rules and even tries to assign values to the variables but fails to solve the problem. In contrast, during the post-test, he identifies the product *xy* as a single entity and makes an explicit substitution in order to find the solution using the general formula (see Figure 4). Here Edwin performs actions such as those described in b) and c).

It is noteworthy that the substitution technique was not explicitly taught to the students in any of the activities of their experimental session with the EM: Nevertheless, this technique is included in EM (that is, substitution is the process used by the machine).

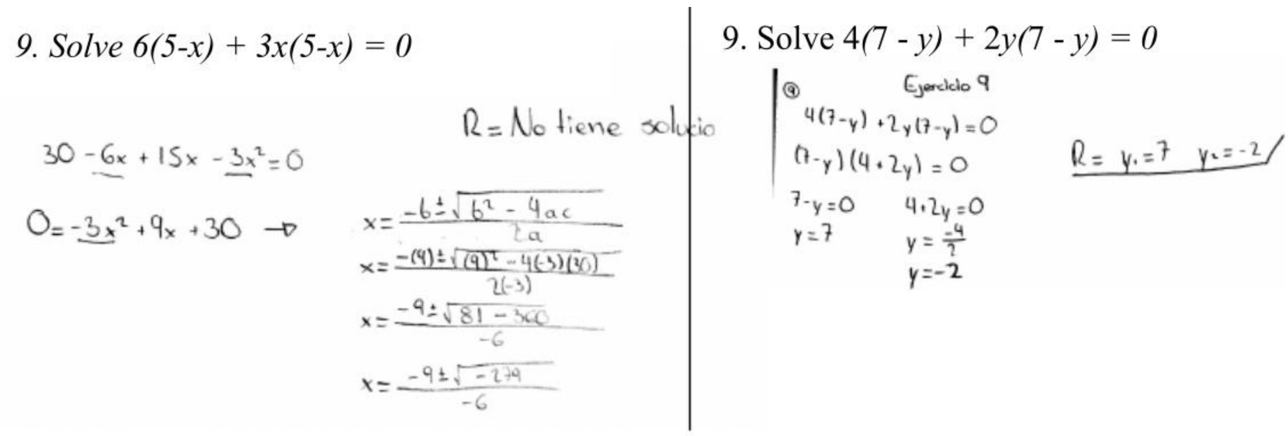

Figure 3. Problem 9, (Edwin *pre-test*/Edwin *post-test*)

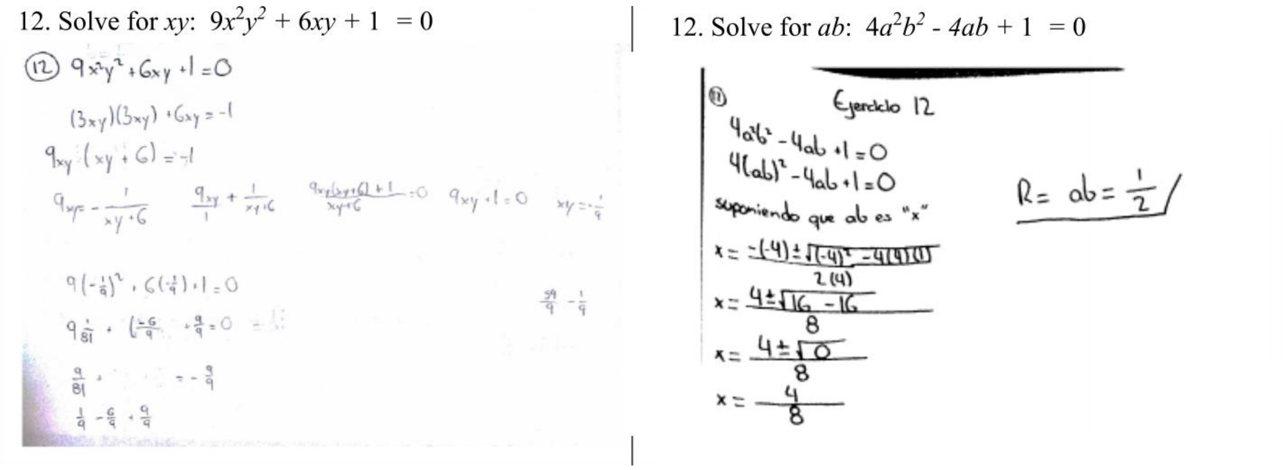

Figure 4 Problem 12 Edwin. (Edwin *pre-test*/Edwin *post-test*)

Results from the Interactive Experimental Session with the EM

The experimental session with the EM lasted for approximately 90 minutes and 4 groups of data corresponding to 3 activities were gathered: 2 answer sheets on the functioning of the machine prior to using it (activity 1), and 10 minutes using the machine (activity 2); video recording of on-screen interactions (activities 2 and 3); and audio clip of the group solving exercises with the EM.

Activity 1: Consisted of projecting the EM on a screen and asking: What do you think the machine is doing? and, how does it work? Three students mention substitution in their answers, although in some cases implicitly. David, Jenifer and Bedani use the verbs elaborate, substitute and assign respectively. Only Bedani answered the question about how the machine works and she did so correctly. *Activity 2:* The EM is designed to not require a manual in order to learn how to use it. Instrumentation is achieved through *affordances*, *feedback* and progressively encountering levels. Therefore, the second activity consisted of a 10-minute-long free exploration. Here it was observed that, in less than 10 minutes, not only did the students learn to use the machine, but they also obtained a fairly accurate idea of its functioning. On average, it took 1 minute and 44 seconds to correctly solve the first exercise. *Activity 3*: 40 minutes of free exploration of the EM. The analysis was centered around the *breakdowns* of student interactions with the machine. Three types of *breakdowns* were identified: one was associated with the process/object duality and the other two were associated with feedback.

Exercise 3/22 of the *Find Input (FI)* scene is analyzed below. Although this exercise is simple, it's challenge lies in the generating expression being a sum and the output expression being a product, that is, they don't have the same structure. This implies that a solution requires for the product *xy* (output expression) to be considered as a single entity. While six of the seven students attempted a solution, only three of them managed to identify the product *xy* as a single entity on the first try, two did so after several attempts, and one was trapped in an operational or procedural conception.

Fabiola's case illustrates a quick transition to identifying the expression as an object. She failed to solve exercise 3/22 on the first try, probably because she didn't see *xy* as a single entity (Figure 5a) and then she abandons the problem. At 17:53 however, she returns; writing down something at minute 19 (Figure 5b). Upon isolating the variables, she clearly parses *xy* as a process and not as an object. On this try, she fails to recognize *xy* as a single entity. However, she offers a correct solution at the end of minute 21 (Figure 5c).

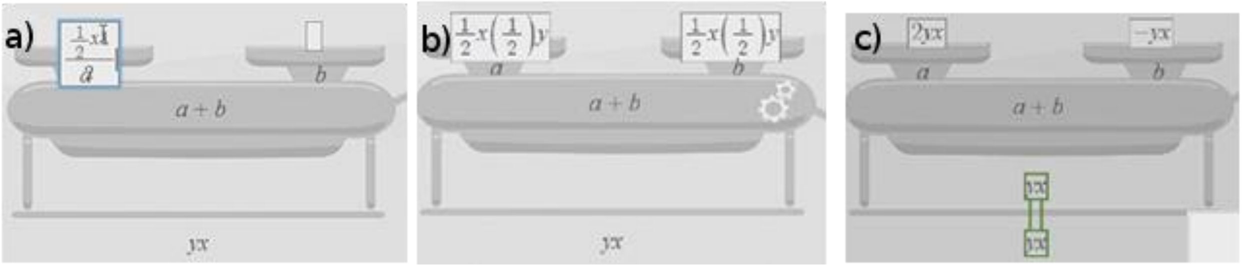

Figure 5 Extracts of Fabiola's Interaction Exercise 3/22 of the FI

This is Jennifer's interaction with exercise 20/22 of *Find Input (*Figure 6). It can be observed that she is unable to solve the problem, possibly due to her trying to guess the answer following the visual salience of the *output.* Jennifer tries to combine sub-expressions of 8x(x+1) to solve it but fails to provide a correct answer.

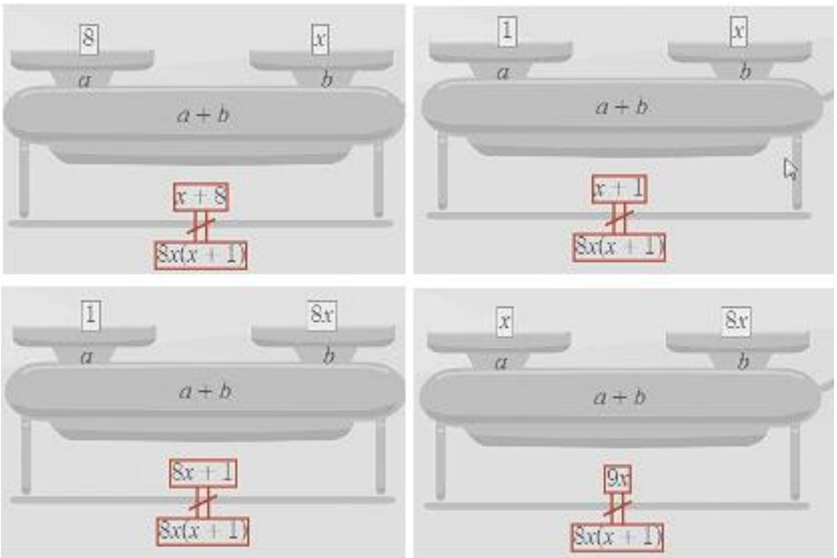

Figure 6. Extracts of Jennifer's Interaction Exercise 20/22 of the FI

The expected solution was to transform the product $8x(x+1)$ into the sum $8x^2+8x$ and put each term on a plate; or rather to identify $x(x+1)$ as a single entity and put a multiple of the expression on each plate so that they can be added to get $8x(x+1)$ (It is noteworthy that other exercises of the *output* scenario already show this as a sum, which simplifies the task).

CONCLUSIONS AND FINAL COMMENTS

The results of the analysis of students interaction with EM suggest a positive answer to research question 1. First, as the EM is essentially an algebraic substitution machine, it favors users adopting that technique. However, the activities proposed also require other actions, such as structure recognition, switching between various forms of an expression or application of known algebraic identities. This is noticeably seen in Edwin's case, who spontaneously applies the change of variable technique to solve two post-test problems. Or Fabiola who, after seveal failed attempts, is able to see product *xy* as an entity and correctly solves the problem. In this sense, and given that the students showed improvement in their structure sense as a group, we can say that the EM facilitates and fosters development of algebra structure sense, where the latter is conceived in terms of actions a)-d).

Regarding research question 2, the HCI aspect of Activity Theory suggests that minimalist design, *affordances*, *feedback* and *level-up* (task design by levels of complexity) had a positive usability effect on students. This is evidenced and confirmed by the nearly null instrumentation period required. In addition, students continued to explore and solve the exercises without intervention from the teacher or researcher.

Level-up is one of the distinctive features of EM, and the results of this research suggest that tasks involving a greater level of difficulty and complexity should be included in the future, in order to trigger stress and *breakdowns* (Bodker) and in turn expand students' learning experience.

The experimental work showed that visual salience often causes students to solve the tasks quickly, without carrying out a structural analysis of the algebraic expression (an analysis based on declarative rules, according to Kirshner). This motivates the inclusion (in a future EM version) of less visually salient levels that require greater cognitive demand from students.

Notes

1. The term *affordances* is used in the sense of Norman (2002), that is to say as suggestions or invitations (of the artifact) for usage possibilities. The broad and general meaning of *feedback* is adopted as 'information provided by an agent (e.g., teacher, peer, book, parent, self, experience) regarding aspects of one's performance or understanding' (Hattie & Timperley, 2007, pg. 81). In the particular case of technology learning environments, the agent is the computer program, which provides feedback to the learner's performance based on its ability to interact with the latter.

2. The design and development of EM was funded by Conacyt – Mexico (V. Munoz-Porras doctoral dissertation, 2015). We want to thank students and authorities of CCH Vallejo school for the facilities to carry out the experimental work of the study.

3. A *serious* game is a game designed with a purpose other than pure entertainment. Its design explicitly emphasizes the added pedagogical value of fun and competition (Wikipedia, consulted on February 22, 2017, https://en.wikipedia.org/wiki/Serious_game).

4. Bødker (1996, pg. 6) uses the term *breakdown* when the learning activity is interrupted because something did not happen as it was expected to (for example, if a button is pressed, but nothing happens). That same author uses the term *focus shift* when interruption of the activity is more deliberate and does not necessarily happen due to a system failure, for instance when the teacher wants to explain something in particular about the operation of an artifact. A *breakdown* is the perception of a discrepancy between our expectations and what actually happens in the world. A breakdown causes a focus shift from the object of the activity mediated by the artifact to the artifact itself.

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