LOOKING AT COMPOSITIONS OF REFLECTIONS IN A DGE FROM THINKING MODES AND SEMIOTIC PERSPECTIVES

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The aim of this paper is to investigate whether and how three modes of thinking and semiotic perspectives are compatible for researching the teaching and learning of elementary geometry in a dynamic geometry environment (DGE). It first provides an epistemological analysis of compositions of reflections in a line from geometric, analytic and abstract aspects. Then, it represents a design of a task considering semiotic potential of particular tools in the DGE that was field-tested with a pair of prospective primary school teachers. Further, it discusses how has the double analyses allowed a detailed understanding of the semiotic potential of the designed artefact for the development of all three modes of thinking of the chosen geometric concept for prospective primary school teachers. It finalizes with suggestions for future investigations of development of knowledge of other concepts in geometry through the modes and their support by digital tools.

Keywords: Composition of reflections, DGE, Three modes of thinking, Semiotic perspective, Integrating technology.

INTRODUCTION

One of the themes of the ICTMT 13 refers to mathematics teachers' education and professional development involving the use of technologies. The selection of the most appropriate content for such programmes is not always a trivial task. It not only has to consider local school curricula requirements but also the enhancement of the learning of mathematics itself by bridging different educational levels systematically. Such systematization necessitates deep insights into epistemological and historical evolutions of mathematical concepts, besides the didactical aspects.

This study tries to bring a possible systematization specifically for the concept of congruence transformations into focus of the analysis. We have considered that the theoretical framework for different modes of thinking of mathematical concepts (Sierpinska, 2000) may be suitable to facilitate our aim. In addition, a creation of digital materials considering the semiotic perspective according to Bartolini Bussi and Mariotti (2008) that may support such structured approach brings innovation and opens new questions not only about the efficiency of the suggested teaching materials but also about the effectiveness of linking these chosen theories in analyzing it. Therefore, we consider the following research question. Are the thinking modes and semiotic perspectives compatible for researching the teaching and learning phenomena of concepts in geometry, e.g. composition of reflections in a line, when they take place in a DGE? Along this direction we present results coming from a case study conducted with two prospective teachers.

THEORETICAL FRAMEWORK

In this work, as already announced above, we refer to two theoretical constructs: first, three modes of thinking of concepts in linear algebra (Sierpinska, 2000) which are to be adjusted for the purposes of geometry, and second, theory of semiotic mediation (Bartolini Bussi & Mariotti, 2008). We argue that these two theoretical frameworks may be used for constructing appropriate theoretical foundation for explanations of the teaching and learning elementary geometry; so first we explain each of them.

Epistemological Considerations

As Winter (1976, p. 16) expressed "symmetry and congruence mappings are considered a fundamental idea in the teaching of geometry even in primary school from several aspects: shape, algebraic, esthetical, economic-technical and arithmetical". Besides, learning complexity of symmetry and rotation notions have also been investigated by researchers (Turgut, Yenilmez, & Anapa, 2014; Xistouri & Pitta-Pantazi, 2006) in different school levels. In order to investigate such phenomena, we hypothesize that three thinking modes that are highly relevant for the research into the teaching and learning of linear algebra could also be useful for studying the development of students' conceptual understanding in geometry, we have chosen to focus on congruence transformations on a plane, in particular, reflections in a line. An appropriate accommodation of the theoretical constructs about the different modes of description and thinking of concepts in linear algebra into geometry is not straightforward.

Congruence Transformations through the Lenses of Thinking Modes

In this paper we focus on isometries, or congruence transformations of the n-dimensional Euclidean Space, particularly, for n=1,2,3. The types of isometries, e.g. for n=2, $E(2)$ are the identity transformation, translation, rotation about a point, reflection in a line and glide reflection. Every isometry of the Euclidean plane is a bijective distance-preserving map. Two geometric figures are congruent if there exists an isometry, which maps one into the other one, that is: either a rigid motion (translation or rotation), or a composition of a rigid motion and a reflection. Let us propose thinking modes in relation to those expressed above.

Synthetic-Geometric Mode (SGM)

In grades 1 to 4 primary schools, Euclidean plane isometries are generally studied typically with the apparatus of geometry. Starting from observing and discussing in- and out of school contexts, through paper folding and drawing, constructing with straight edge and pair of compasses, pupils gain knowledge about some of the distant-preserving transformations. In this period, usually, due to the level of mathematics, no explicit reference to $E(1)$ or $E(3)$ is made. Mathematical objects such as points, lines, planes or triangles refer to the SGM. In other words, SGM is also considered as a kind of 'thinking in-action' (Sierpinska, 2005), i.e., thinking about the objects in coordinate-free geometry, but not about how they are constructed on. Consequently, if a student speaks about geometric objects, for example, points, lines, triangles or basic properties of them, then, those are traces of SGT mode.

Analytic-Arithmetic Mode (AAM)

While geometrical approaches for the introduction to the congruence mappings in school are widely accepted, the analytic-arithmetic mode of thought, though being an inseparable part of the concept, often remains unnoticed. The analytic counterpart that relates to the use of arithmetic language and symbolism is rarely conducted even in lower secondary school mathematics. While drawing, sketching and visualizing refer to the SGM and are typical school activities, thinking of congruence mappings as functions (from the plane in itself) in an analytic-arithmetic mode, persists out of the scope in school. Within the context of elementary geometry, representing objects as a system or using formulas to describe the action can be considered as a kind of AAT mode.

Analytic-Structural mode (ASM)

The set of isometries of the Euclidean plane $E(2)$ with the operation composition of functions forms a group (closure, associative, identity and invertibility properties). A glance on the historical evolution shows that the Euclidean groups E(n) of n—dimensional Euclidean space are among the oldest and most studied, at least implicitly for n=2,3, long before the concept of group was introduced. This historical geometrical conduction, prior the algebraic and the abstract, seems to be reflected in mathematics school curricula and textbooks designs even today. In primary schools, the abstractness is largely decreased. Yet, in our opinion, this knowledge is also relevant for teacher education and teacher professional development programs. Ignorance of any of the modes may prevent pupils from further earlier cognitive development. For an illustration, incomplete pupil's acquisition of reflection in a line in grade 3 may occur as a result of a teacher's insufficient personal resources about reflection regarding components of teacher's knowledge as reported by Donevska-Todorova (2016). "Interestingly, students mostly do not use symmetry to explain a particular conjecture" (De Villiers, 2004, p. 713) about a geometric figure (e.g., isosceles trapezoid) by dragging even in cases when it has been constructed by means of line reflection. Both prospective and practicing mathematics teachers usually require substantial assistance with the formal defining (e.g. of an isosceles trapezoid) but they do indeed develop abilities of descriptive and constructive defining (De Villiers, 2004, p. 722).

With respect to traces of AST mode can be considered as emergence of thinking about mathematical objects and conjecturing about the *action*, and/or making generalizations about the mathematical properties. For example, within the scope of this paper, thinking about congruence and group of functions such as identity function, i.e., inverse, associative and other properties of the function can be considered as traces of AST mode.

Theory of Semiotic Mediation (TSM)

The TSM proposed by (Bartolini Bussi & Mariotti, 2008), not only aims to construct mathematical meanings in a social communicative environment (where the teacher has a role of a mediator), but also to analyse teaching-learning process with a semiotic lens. In the TSM, in mediation process, the teacher focuses on specific artefacts and intentionally but carefully uses them to guide students' personal meanings to desired, culturally accepted mathematical meanings. At the same time, the teacher analyses possible evolution of signs that foster students' learning. Consequently, TSM bases on two key notions: (i) *semiotic potential* of an artefact and (ii) design of *didactic cycles*. The first refers to epistemological and didactical analysis of the artefact's evocative power to stimulate emergence of meaningful mathematics (Mariotti, 2013), while the second refers to (carefully) design the teaching-learning environment, specifically in the light of the epistemological learning route elaborated in the first phase.

A complex semiosis could be observed when the students interact with the artefact. In order to classify the signs that emerge, Bartolini Bussi and Mariotti (2008) have identified three type of signs: *artefact signs* (AS), *mathematical signs* (MS) *and pivot signs* (PS). AS immediately emerge when the student uses the artefact, and they are generally in relation to practical observations, specifically about the artefact. MS refer to mathematical meanings that are accepted by the community by generalizing and/or expressing a conjecture, a definition or a proposition. PS underline interpretative link between personal meanings and MS sometimes including hybrid expressions.

METHODOLOGY

The participants of this case study are two (sophomore level) prospective teachers (A, B, both nineteen years old females) from a department of primary education. Regarding the mathematical content, the students had experience mainly in algebra, e.g. relations, functions, and (*2D* and *3D*) geometry, e.g. geometric transformations and their representations and notations as functions (independent-dependent variables, etc.). However, they did not have any experience with compositions of reflections. Regarding didactical considerations, the participants had experience with a dynamic geometry system (DGS), e.g., GeoGebra. They were familiar with fundamental tools, their roles and distinctive property of any DGS: initial drawings (independent objects) can be dragged but constructed (dependent) objects cannot.

Task-based interviews were video-recorded, where screen-recorder software worked synchronously. The interview lasted half an hour and the collected data coming from two videos and students' productions were analysed through a double lens, first referring to the thinking modes, and second, from a semiotic lens.

Semiotic Potential of Specific Functions and Tools of a DGS and the Task

The mathematical context embedded in aforementioned three thinking modes in elementary geometry provided us to consider *compositions of reflections* on the Euclidean plane $\mathbf{R}^2$ in a specific DGS GeoGebra. We have considered scalene triangles and compositions of two reflections in a line. First, a triangle ABC was reflected (σ_1) according to line l (the black line in Figure 1), by this way obtaining the triangle $A'B'C'$. Next, the triangle $A'B'C'$ was reflected (σ_2) according to a line g (the purple line in Figure 1) resulting with a triangle $A''B''C''$. Consequently, with respect to the position of the lines, one can refer to three separate cases: (1) when the axes of reflections coincide (Figure 1a), (2) when the axes are parallel (Figure 1b) and (3) when the axes intersect in a point (Figure 1c, 1d).

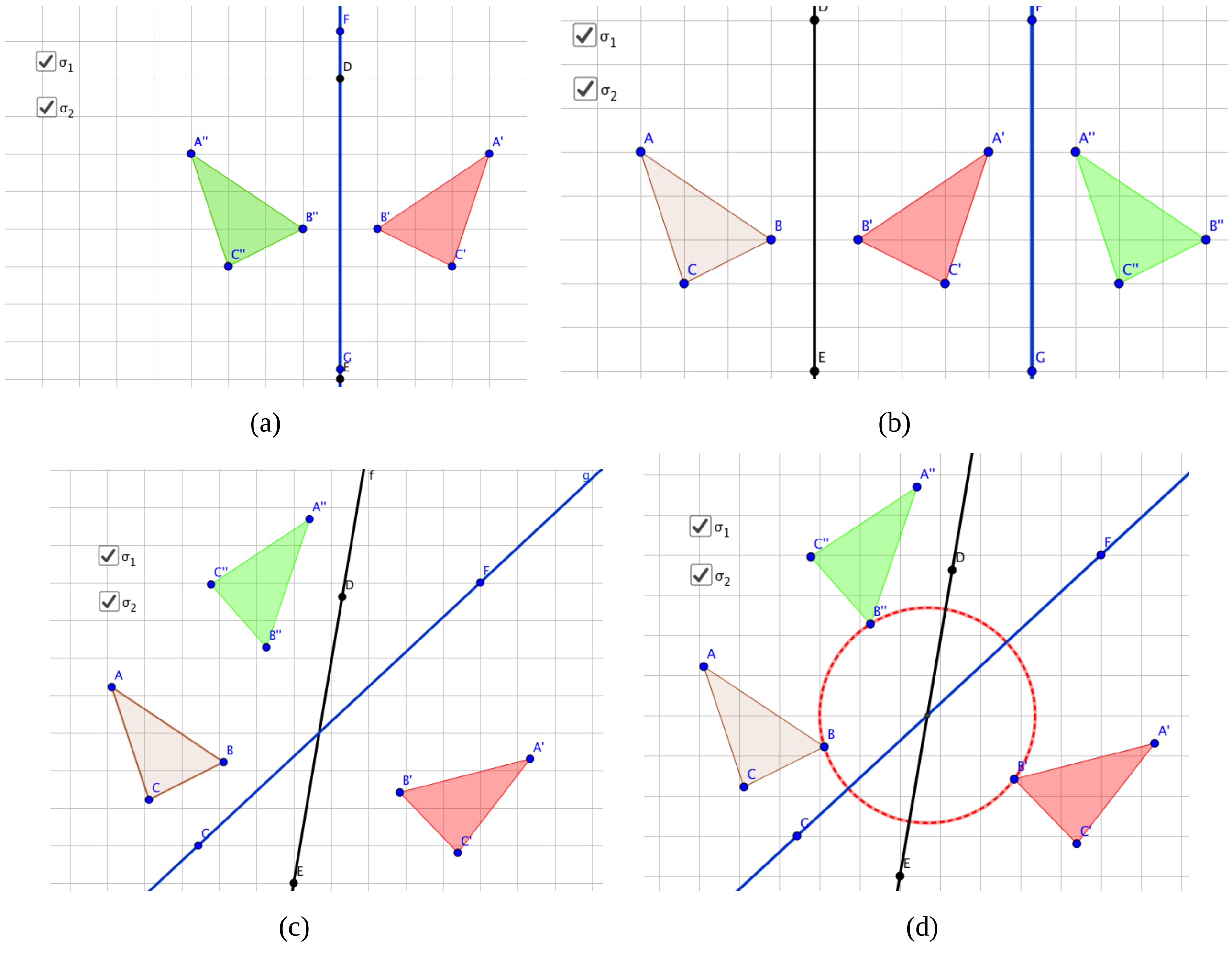


Figure 1: (a), (b), (c), (d) Three cases for compositions of reflections

With the terminology of the ASM, one could implicitly refer to properties of a group, in particular: *closure* (cases on Figure 1b: translation for a vector and Figure 1d: rotation where the intersection point of the reflection lines is the center of the rotation, both being isometries) and *neutral element*, i.e. identity transformation (case in the Figure 1a). We hypothesize that the following tools and functions of the DGS (in this case GeoGebra) have semiotic potential for creating meaning for such cases, i.e., mathematical notions expressed above:

– *Dragging* function enables the students to move and manipulate free (independent) geometric figures, by this way creates an environment for exploring different situations. However, dragging function of any DGS does not work for constructed (dependent) objects, which also provides the notion of co-variance of the objects.
– *Grid* function of any DGS constructs parallel lines on the geometric plane, which can contribute students' observation of distances between initial and reflected objects.

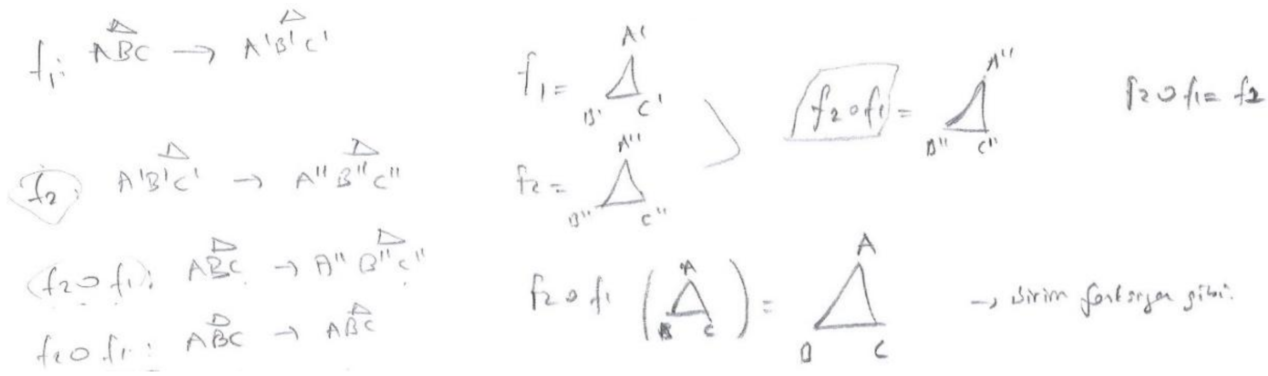
Following this, the task delivered to the students was: Step I: Click on σ_1, drag the points or the line *l*. Explain your observations mathematically. Step II: Click on σ_2 and follow the first step. Step III: Explain the relationships between triangles and generalize your findings.

ANALYSIS WITH A DOUBLE LENS

The discussion started by asking the students to follow steps of the task, what is a composition of two reflections in a line. In the first step, they focused on the reflection in the line *l* and realized that points *A*, *B* and *C* can be dragged. The following excerpt in Table 1 (Unit I) was drawn from this discussion (I: Interviewer), where we also provide first step of a double analysis.

Unit I	Thinking Modes Analysis	Semiotic Analysis
[14] B: … A, B and C can be dragged. Then this triangle [*points ABC triangle*] can be dragged. [15] I: Did you check the other points? [16] A: But it is depended… [17] B: Yes, because this [*points A'B'C' triangle*] is depended on initial one… [18] A: Exactly. I mean there is a transformation here something like that [*writes* $f(\Delta)=f'$]… [19] B: Yes. Something like that. When *x* varies, then *y* varies you know. Nevertheless, here we have triangles as variables … we can write [*writes* f(x)=y, *x* independent]. Because *y* is dependent on *x*, here this triangle [*means A'B'C' triangle*] is depended on the initial triangle [*explains pointing on the window*]. Therefore, we cannot drag this. … [22] A: a reflection transformation… [23] B: Actually, points are transforming. Then triangle is transforming, and then we have a new triangle…	– In [14-17], students speak in-action, i.e., about movements of geometric objects can be dragged on the screen. Actually, they speak about what they observed when they drag the moveable objects. Those are *traces* of being in SGM, although they mention the notion of dependent-independent variables. – In [18-19], the students move forward from SGM to AAM by beginning to use symbolic language of the action. They express the s_1 reflection through function *f* and express independent and dependent triangles on the screen by explaining which can be dragged. – In [22-23], the students characterized transformation and related situation with their pre-knowledge. B's explanation reflects her thinking about transformation as a mathematical object, which can be referred as being in ASM.	- The students' immediate observations are due to the use of the artefact. For instance, not only verbal expressions such that "can be dragged", "this triangle", "is depended on", "depended on", "cannot drag", but also their gestures for pointing triangles can be considered as AS. - There appears a specific PS here: the notion of function, which contributes students to emerge their personal meanings with their observations coming from the artefact. In other words, the PS function mediates the emergence of a mathematical characterization: "… points are transforming. Then the triangle transforming…" that can be considered as a mathematical expression and also a manifestation of MS.

Table 1: Double analysis of the Unit I



(a) (b)

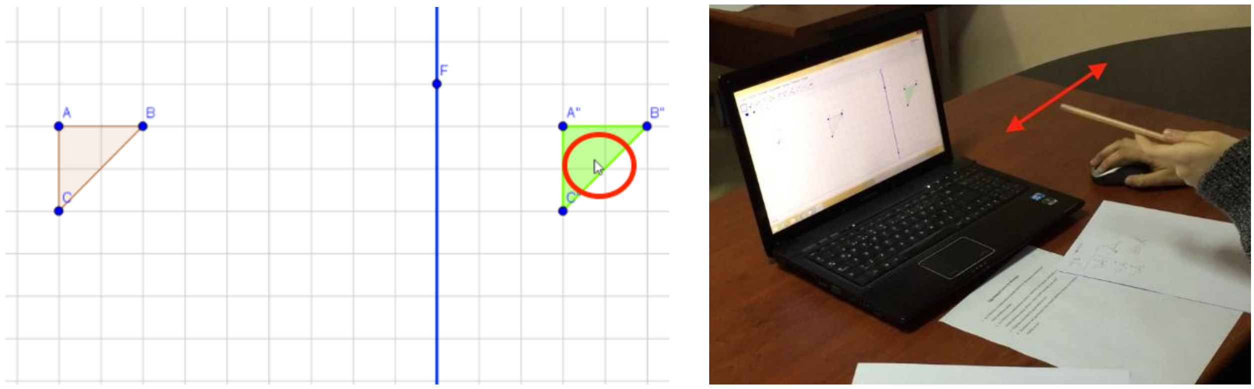
Figure 2: (a) A's mathematical expressions, (b) B's mathematical expressions

Then, the students were asked to follow the second step of the task and express what they observed by clicking s_1 and s_2 reflections. They immediately observed the second reflection with respect to the line *g*. The students used an interesting terminology to represent composition of reflections. The following excerpt (Unit II in Table 2) shows the discussion and the second step of the double analysis.

Unit II	Thinking Modes Analysis	Semiotic Analysis
[35] A: Could you drag line *l*? … The second transformation is also depended on the movement of line *l*. [*B drags the lines*] Actually, this is… [36] B: This [*means the final triangle* $A''B''C''$] is our new dependent variable. [37] A: [*Pointing the* $A''B''C''$ *triangle*] Dependent variable is changed… … [40] I: How can we express this situation mathematically? [41] A: For example, let me show the second transformation with f_2 [*she writes the second row of* Figure 2a] … But, finally we have [*she writes the third row of* Figure 2a]; because of the two composite reflections. [42] B: Yes. [*She writes synchronously to* A, see the first row of Figure 2b].	– In [35-37], the students, again, speak about the movements and draggable points and lines. They characterize "new" dependent – independent variables. However, they are aware that the dependent variable "is changed". – After the teacher's intervention [40], the students immediately relate the compositions of reflections with composite functions. They use mathematical expressions of the composite of reflections [41-42]. These all over imply that the students are in the AAM.	- Students discuss about artefact's feedbacks about dragging (e.g., "drag line *l*", "movement of line *l*"), which were AS. However, their observations trigger to emergence of a new PS: *independent–dependent* variables in the compositions of reflections (e.g., "new dependent variable"). - Finally, they use mathematical representations to express their mathematical meanings (*use of a triangle as a dependent or independent variable* in Figure 2) about compositions of reflections. This can be considered as an example of how AS transform into MS.

Table 2: A double analysis of the Unit II

Further, the discussion continued about the three different positions of the lines, which affect the positions of the initial and final triangles. Therefore, the students were asked to unclick the first reflection and discuss the mathematical situation on the screen (see Excerpt III in Table 3).

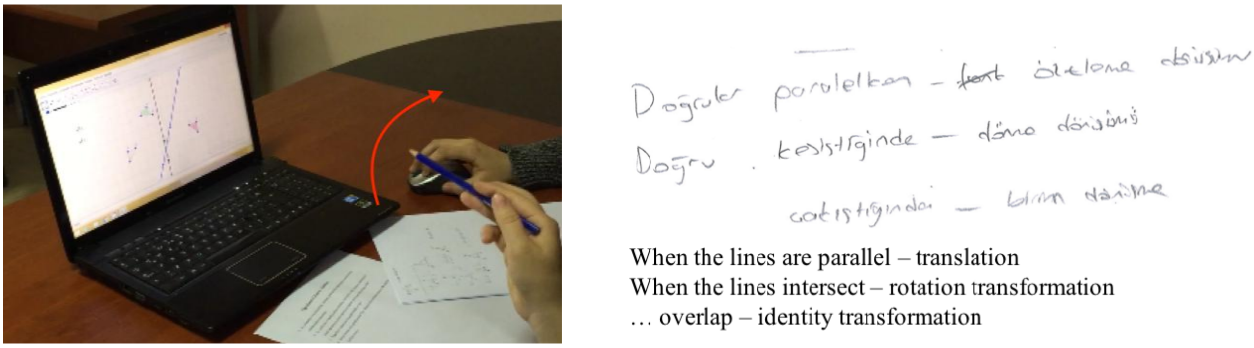


(a) (b)

Figure 3: (a) B's cursor mimics, (b) B's translation gesture with pencil

Unit III	Thinking Modes Analysis	Semiotic Analysis
[45] I: Ok, right. Please unclick σ_1, … what do you observe when you drag *ABC* triangle? [46] B: … This [*mimics with cursor*, see Figure 3a] is not a reflection… The distance …For example, we have *ABC* triangle, but it seems like translated into [*she gestures with pencil*, see Figure 3b] $A''B''C''$ triangle… … [48] I: Ok. Please click on σ_1 … Check the position of such lines! [49] B: They are now parallel… [50] A: They can intersect, either can be parallel and they can overlap. [51] B: Let's move this [*she drags the line l onto line g*]. [52] A: The initial and final triangles overlapped! [53] B: Like functions… [54] I: How can we express this situation mathematically? [55] B: One-to-one and onto …? [*She writes the last line of* Figure 2b]… Is this identity function? … [56] A: … [*She writes the last line of* Figure 2a]. … [67] B: Let's intersect the lines… [*she drags continuously and tries to understand the situation*] [68] A. Here … [*gestures with* pencil, see Figure 4a]. Like a… [*B drags the points and lines*] The final triangle is rotated around initial triangle. Yes this is now a rotation… [*They together write their conclusions*, see Figure 4b]…	– In [46], the student use her spatial perception and therefore express their observations in the case of the axes are parallel, even she finds the translation of the triangles, but not mathematically. Because she does not mention any *translation vector*. Also, in [49-52], the students speak about their observations on the screen, not about mathematical necessities. All those are traces of being in SGM. – However, in [53], B relates the situation with functions. She also realizes that such kind of reflection might be similar to identity function and have one-to-one and onto properties. A also uses a similar notion. Since, suffice it to say that, in this point, they are in AAM, since they does not generalize the situation [55-56], and does not mention how this could be possible. –They finally explore the case when the reflection axes intersect. They analyze the three cases and make a generalization [67-78], which seem a kind of having ASM.	- In [46], mimicking with cursor, "the distance", "*ABC* triangle", "translated into" and also B's gesture with pencil and $A''B''C''$ triangle are AS. - In [49-52], there appear several AS. For example, "parallel", "intersect", "overlap", "move this", and "the initial and final triangles overlapped" are also AS. - In [53] pivot signs "function" and "identity function" and also specific expressions in Figure 2 appear, which show interpretative link with classification of composition of reflections with respect to axes. - They finally categorize the cases, and characterize composition of reflections with respect to positions of axes. They express their conjectures with validation through dragging, e.g., "yes this is now a rotation", which can be accepted as traces MS.

Table 3: A double analysis of the Unit III



(a) (b)

Figure 4: (a) A's rotation gesture, (b) Students' conclusions (translated into English)

FINDINGS AND DISCUSSION

Based on the double analysis in Table 1, we have found out that the PS of the artifact have mediated an appearance of MS related to congruence of triangles in a SGM of thinking. Then, in contrast to our expectations that the most frequent mode would be the SGM, it was the AAM, which dominated indeed. A reason for it may be the students' pre-knowledge about transformations. Yet, the analysis in Table 2 shows that it may also be a result of the semiotic potential of the created DGE to stimulate an emergence of MS due to interactions within the artifact. Further, our double analysis has shown that the occurrence of the ASM, manifested through an axiomatic property of a group, e.g. the identity transformation on the plane (Figure 2a, on the bottom), could be influenced by the potentials of the design (Table 3). An observation of the students' written materials, leads to a conclusion that though the symbolic language is not fully developed, the AAM was influenced by the SGM of thinking (triangles occur as variables on Figure 2) showing that the design has contributed to changes from one into another mode of thinking. This analysis has leaded us to propose a diagram of possible links between the three modes of thinking and the potentials of the DGS for the emergence of the three signs (Figure 5).

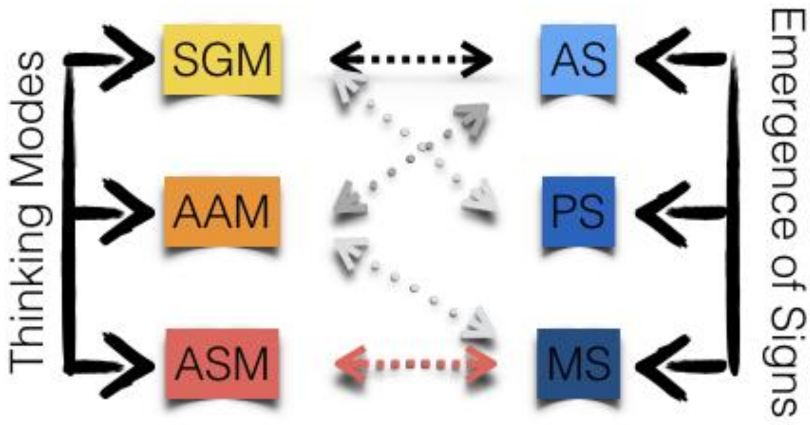


Figure 5: Relationships between the modes of thinking and emergence of signs

CONCLUSIONS

In this paper, we have considered the research question, 'Are the thinking modes and semiotic perspectives compatible for researching the teaching and learning of composition of reflections in a line taking place in a DGE?' Firstly, we have shown that the "borrowed" terminology related to the three modes of thinking of concepts in linear algebra (Sierpinska, 2000) may be meaningful for studying the teaching and learning processes of certain concepts in elementary geometry, in this case, congruence of reflections in a line. Additional exemplary geometric concepts are required in order to investigate whether (or to which extent) a “nested diagram” of three modes of thinking in linear algebra (Donevska-Todorova, 2017) is also suitable or adjustable for studying concepts in geometry on a local level.

Secondly, the analysis from a semiotic perspective has shown how students’ personal meanings transformed into mathematical meanings, and how gestures contributed to emergence of students’ thinking. One interesting point in the semiotic analysis was how gestures contributed students’ thinking and emergence of mathematical meanings. Another point was about affirmative result of the semiotic potential of the specific functions and tools of a DGS, which confirmed that those could be considered as a tool of semiotic mediation that is consisted within the recent literature (Turgut, 2015, 2017).

Finally, our double analysis has provided affirmative insights into existing relationships between the three modes of thinking and the potentials of the tools of the designed DGE (Figure 5). As seen from Figure 5, SGM, sometimes were in relation to both AS and PS, while AAM mode also implied both AS and MS. But interestingly, ASM separated from AS and PS and was directly in relation to MS. Complexity of analysis tools of the thinking modes and semiotic perspectives also appeared in a recent study with respect to learning the notion of parameter in linear algebra (Turgut & Drijvers, 2016). We express our awareness of the affordance and limitations of this diagram for interpreting explicit relationships between the modes and the semiotic potentials by pointing out that such confirmations require further research.

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