(Un)intended representations in dynamic geometry software: pedagogical considerations

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In recent years, dynamic geometry software (DGS) has become common in classrooms for teaching and learning of mathematics. In this paper, I address some representational issues with which students and teachers may encounter while using DGS. Unintended representations may stem from the design principles for DGS, tasks that involve constructions with a limitation and representations of mathematical objects in DGS. Pedagogical considerations about using those representations as an opportunity for mathematical investigation are discussed.

Keywords: dynamic geometry software, (un)intended representations, pedagogical considerations

INTRODUCTION

Mathematics educators call for using technology in mathematics classrooms such as dynamic interactive mathematics technologies (Association of Mathematics Teacher Educators (AMTE), 2006; National Council of Teachers of Mathematics (NCTM), 2000). An issue for teachers may include finding the right tool to use in the mathematics classroom to enhance students' mathematical learning (Smith, Shin, & Kim 2016). Smith et al. (2016) emphasize that a quick search on the Internet for a mathematical topic yields in a number of commercial and free-of-charge tools.

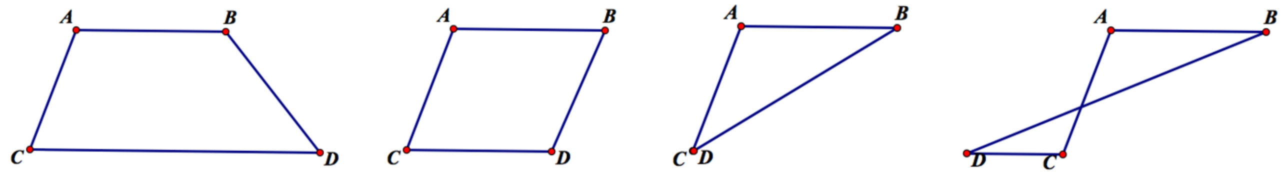
Dick (2008) provides some criteria for selection of technologies teachers may take into consideration. For example, a technological tool should stay true in mathematics – that is known as *mathematical fidelity* (Dick, 2008; Dick & Hollebrands, 2011). Also, a digital tool should not trigger a mismatch between students thinking and intended mathematics learning – that is known as *cognitive fidelity* (Dick, 2008; Dick & Hollebrands, 2011). For example, the angle between two perpendicular lines is perceived as an acute or obtuse angle in an unequal scale of coordinate system (see Dick & Burrill, 2016; Dick & Hollebrands, 2011). Moreover, a technological tool should be pedagogically faithful, in that "the student should perceive the tool as (a) facilitating the creation of mathematical objects, (b) allowing mathematical actions on those objects, and (c) providing clear evidence of the consequences of those actions" (Dick, 2008, p.334). Smith et al. (2016) found that pedagogical and mathematical fidelity for selecting a digital tool to use in classrooms was important for in-service and prospective mathematics teachers value.

Leung and Bolite-Frant (2015) emphasize that a technological tool with a limitation or uncertainty has a *discrepancy potential*. The researchers state that unintended mathematical representations open a pedagogical space for teachers. For example, teachers may capitalize an unintended mathematical concept with a focus on technological representations. A pedagogical space may take place by means of "feedback due to the nature of the tool or design of the task that possibly deviates from the intended mathematical concept or (ii) uncertainty created due to the nature of the tool or design of the task that requires the tool users to make decisions" (p.212). In this paper, representational issues stemming from constructions with a limitation, the design of DGS and representations of mathematical objects in DGS are discussed.

CONSTRUCTION WITH A LIMITATION

DGS allows for manipulating primitive elements (e.g., points, line segments) and exploring the invariant attributes of geometric objects. Properties of geometric objects remain invariant when a point or object is dragged in a properly constructed shape (Laborde, Kynigos, Hollebrands, & Strässer, 2006). Students or teachers may use a DGS *drawing* – that is "a process that involves the use of "freehand" tools to create a geometrical object", and focus on its perceptual characteristics (Hollebrands & Smith, 2009, p.221). Also, teachers may provide a construction with a limitation for students.

Drawings or constructions with a limitation trigger unintended mathematical representations (see Mariotti, 2013; Ruthven, Hennessy, & Deaney, 2008). Such technological representations stem from how the tools in DGS are utilized. For example, in Figure 1a, the *Parallel line* tool is utilized to create a trapezoid with one pair of parallel sides. Students may notice a trapezoid can also have two pairs of parallel sides dragging point *C* towards point *D* (Figure 1b) and conclude that "a trapezoid is sometimes a parallelogram." When points *C* and *D* coincide as shown in Figure 1c, the trapezoid becomes a triangle. Moreover, if point *C* crosses point *D*, a crossed quadrilateral is created (de Villiers, 1994). This construction does not preserve the invariant properties of the trapezoid and has a limitation. However, this construction may allow for a mathematical discussion about the counterexamples of the trapezoid. Researchers point out that constructions with a limitation or drawings give an opportunity for students to reason about geometric objects with the supervision of mathematics teachers. For example, Ruthven et al. (2008) stress that teachers capitalize drawings or unintended mathematical constructions. On the other hand, they observe a teacher who concealed anomaly constructed geometric objects.



(a) (b) (c) (d)

Figure 1. (a) A trapezoid, (b) The trapezoid becomes a parallelogram, (c) The trapezoid becomes a triangle, (d) The trapezoid becomes a crossed quadrilateral

Unintended representations may result in interruptions in the flow of a lesson and teachers may make ad hoc decisions about how to respond in these moments (Cayton, Hollebrands, Okumuş, & Boehm, 2017). For example, teachers may emphasize counterexamples of geometric objects using an unintended representation with a focus on mathematics and technological representation. On the other hand, they may eliminate unintended representations. Teachers' pedagogical dispositions determine if they conceal, capitalize or eliminate an unintended mathematical representation (Dick & Burrill, 2016; Mariotti, 2013; Ruthven et al., 2008). For the elimination of unintended representations, the teacher should use his or her technological and mathematical knowledge to construct objects that stay true in mathematics (Dick & Burrill, 2016). For example, the restriction of point *D* on a ray that is parallel to $\overline{AB}$ as shown in Figure 2a eliminates the counterexamples of the trapezoid. Then, point *C* does not meet at or cross point *D* (Figure 2b). Teachers' mathematical and technological knowledge should be in action to construct a geometric sketch that preserves the critical attributes of a geometric shape (Dick & Burrill, 2016).

Figure 2. (a) Point *D* bounded on a ray, (b) Points *D* and *C* do not coincide

DESIGN OF DGS

Developers of DGS make design decisions and users (e.g., teachers) most often have no freedom to change the interface for the tool. The interface for a tool may violate mathematical fidelity and provide incorrect feedback (Dick, 2008; Dick & Burrill, 2016; Dick & Hollebrands, 2011). In GeoGebra (a free dynamic geometry program), the *Angle Bisector* tool creates two angle bisector lines when two lines/line segments are selected (Steketee, 2010). Then, angle bisectors of a triangle meet at four points as shown in Figure 3a. This representation may be confusing for students and teachers because three angle bisectors of a triangle should meet at a point – that is called *incenter*. However, the design decision on the *Angle Bisector* tool results in demonstrating the excenters of a triangle [the center of a circle that is tangent to a side of a triangle and the extension lines of the other two sides]. Teachers may prefer to use DGS that provides more transparent feedback for students. For example, the Geometer's Sketchpad (a commercial dynamic geometry program) creates a ray as an angle bisector and the angle bisectors of a triangle meet at a point (see Steketee, 2010).

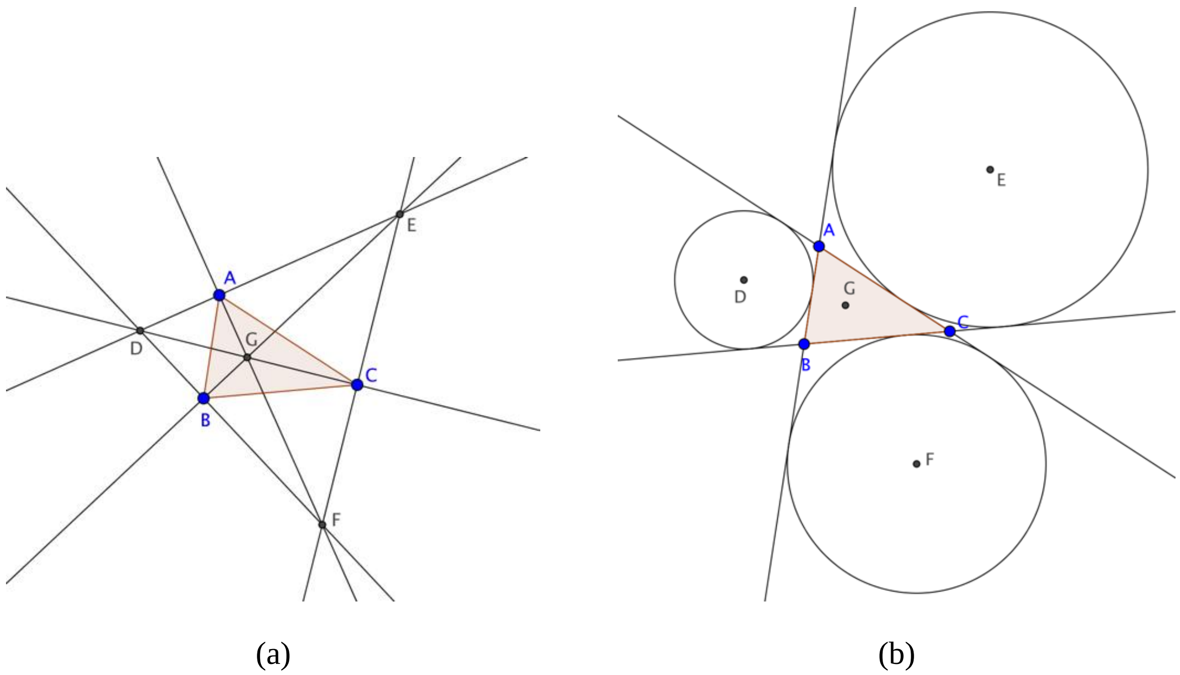
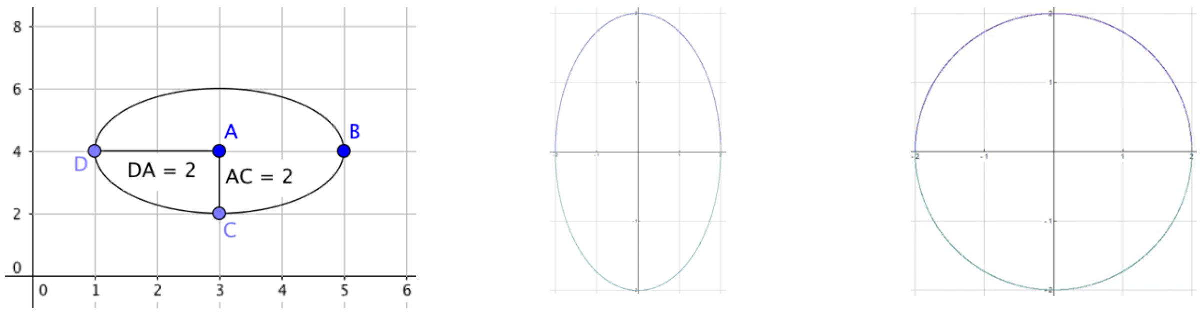


Figure 3. (a) Angle bisectors of a triangle in GeoGebra, (b) The excenters (Points *D*, *E*, *F*) of a triangle

Teachers should be able to make sense of an unintended technological representation to determine if the tool provides a correct mathematical representation. This skill requires establishing a link between mathematical and content knowledge to reason about unintended representations (Dick & Burrill, 2016). Teachers may utilize different techniques to use the tools in DGS to eliminate unintended representations. For example, the *Angle bisector* tool in GeoGebra does not demonstrate the excenters if the three vertex points of a triangle are selected. Knowledge of an alternative utilization of a tool in DGS or about different dynamic geometry programs may assist teachers in making a decision about identifying the right DGS.

Smith et al. (2016) found that in-service and student teachers were not concerned about cognitive fidelity. However, tools that violate cognitive fidelity result in giving misleading information (Dick & Burrill, 2016; Dick & Hollebrands, 2011). For example, in GeoGebra, one may change the scale of coordinate system. As shown in Figure 4a, a circle in an unequal scale of coordinate system looks like an ellipse (see Steketee, 2010). On the other hand, some programs (e.g., Graphic Calculus) create graphs on an unequal scale of system as default when a graph is plotted (Figure 4b). Familiarity with the tool may eliminate unintended representations. For example, the *Square* tool in Graphic Calculus equalizes the axes (Figure 4c) (van Blokland, van de Giessen, & Tall, 2006).



(a) (b) (c)

Figure 4. (a) A circle on an unequal scale of coordinate system in GeoGebra, (b) a circle in an unequal scale of coordinate system in Graphic Calculus, (c) the circle in an equalized coordinate system

REPRESENTATIONS OF MATHEMATICAL OBJECTS IN DGS

According to Laborde (1993), "*drawing* refers to the material entity while *figure* refers to a theoretical object" (p.49). *Material drawings* (e.g., diagrammatic representation of a circle in DGS or on a sheet of paper) have flaws, for example, "marks have a width, straight lines are not really straight" (p.50). She refers to the abstraction of material drawings as *idealized drawings*. A material drawing may result in a confusion for students/teachers and they may have difficulty identifying its corresponding figure (theoretical object). Similarly, a dual relationship between mathematical and technological representations should be established because DGS may not provide an accurate representation for a figure. For example, a quadrilateral signifies a plane in Cabri 3D as shown in Figure 5. Then, students may think of a plane as a quadrilateral or a bounded object because it does not extend in all directions forever. The *Sector* tool that extends the plane as shown in Figure 6 may be utilized to demonstrate the unboundedness of plane. Knowledge of tools in DGS assists teachers in providing a more accurate representation of plane. Teachers may consider using different dynamic geometry programs to develop their understanding of figures. For example, some dynamic geometry programs (e.g., GeoGebra) do not allow students/teachers to extend the plane in all directions. Google SketchUp provides a more accurate representation of plane as default (see Panorkou & Pratt, 2016). Then, dynamic geometry programs have different discrepancy potentials.

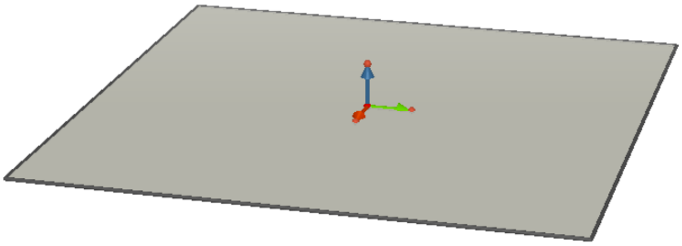


Figure 5. Representation of a plane in Cabri 3D

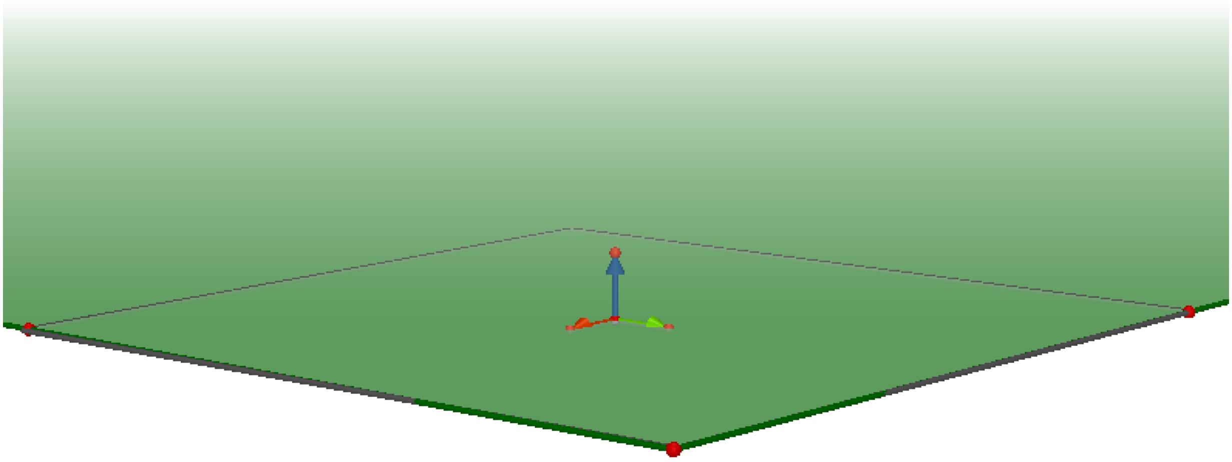


Figure 6. Extended plane in Cabri 3D

PEDAGOGICAL CONSIDERATIONS

Dynamic geometry programs that violate mathematical, cognitive and pedagogical fidelity may disrupt the flow of a lesson if teachers do not pre-plan to use them. Representational issues may stem from the design principles for DGS, tasks that involve constructions with a limitation and representations of mathematical objects in DGS. Researchers emphasize the importance of teachers' abilities in identifying affordances and constraints of a tool with a focus on how a tool may help or hinder students' thinking (Bartolini Bussi & Mariotti, 2008, Leung, & Bolite-Frant, 2015).

Dick and Burrill (2016) address that technological content knowledge "is important for teachers employing technology in the classroom, for it can help them anticipate what issues and phenomena students may encounter while using technology for a mathematical problem solving task or exploration" (p.44). For example, teachers may eliminate a representational issue and provide students accurate representations or constructions using their technological content knowledge. Also, knowledge of different dynamic geometry programs may guide teachers through technologies that have the best potential to enhance students' learning. Accordingly, they may prefer to DGS that is pedagogically, cognitively and mathematically faithful.

On the other hand, how a teacher makes an ad hoc decision when they encounter with an unintended representation is related to their technological pedagogical content knowledge. On the one hand, they may conceal unintended representations (Ruthven et al., 2008). On the other hand, they capitalize unintended representations with a focus on technology and mathematics (Mariotti, 2013; Ruthven et al., 2008). For example, Mariotti (2013) found that a student's drawing that did not preserve the invariant properties of a square gave an opportunity for students to construct a square using the function tools of DGS (e.g., the *Perpendicular Line* tool). In other words, the teacher used the student's drawing as an opportunity for mathematical investigation and generated a whole-class discussion. Also, inaccurate representations may allow students and teachers to revisit the definitions of a geometrical object and discuss about its counterexamples. Leung and Bolite-Frant (2015) emphasize "task design can intentionally make use of a tool's discrepancy potential to create uncertainties and cognitive conflicts which are conducive to student learning" (p.221).

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