

STUDENTS' EXPANDING OF THE PYTHEGOREAN THEOREM IN A TECHNOLOGICAL CONTEXT

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Dynamic Mathematical Software (DMS) in general and GeoGebra in particular have attracted the attention of mathematics educators because of their potential to influence student learning. The present research aims to add to the growing research efforts to study the influence of GeoGebra on processes used by dyadic learners to construct knowledge. Specifically, we study the context of three pairs of seventh graders who worked on an inquiry task to construct the relations between areas of squares built on the sides of an obtuse/acute triangle in a specially designed GeoGebra environment. The analysis used the Abstraction in Context (AiC) framework. The findings indicate that the three pairs constructed all the expected knowledge elements, and that one pair constructed unexpected construct. Generally, findings indicate the positive influence of the GeoGebra technological tool on the construction processes.

Keywords: GeoGebra, Construction of knowledge, Pythagorean theorem

INTRODUCTION

Two areas are at the core of the current study: (1) the construction of abstract mathematical knowledge; and (2) the use of dynamic technology in mathematics education. These two domains are examined below.

Construction of abstract mathematical knowledge

Understanding how learners construct abstract mathematical knowledge is a central aim of research in mathematics education. Abstraction in Context (AiC) is a theoretical framework for describing processes of abstraction in different contexts (Dreyfus, Hershkowitz, & Schwarz, 2001). Dreyfus et al. (2001) defined abstraction as a process in which previous mathematical constructs are vertically reorganized into a new structure. The role of context is central to the process of constructing abstract mathematical knowledge. Several contextual factors may influence mathematical abstraction, including the students' prior knowledge, the nature of the task, and interactions with other learners and with technology. Understanding the role of context may lead to a better understanding of abstraction processes. Hence, the present study began with a careful design of the contextual factor – technology applet.

The AiC framework postulates that the genesis of abstraction passes through a three-stage process: the need for a new construct; the emergence of the new construct; and the consolidation of that construct. The emergence of a new construct is described and analyzed by the RBC model: recognizing (R), building-with (B) and constructing (C). Recognizing refers to the learner's realization that a previous knowledge construct is relevant for the situation at hand. Building-with involves combining recognized constructs in order to achieve a localized goal, such as the actualization of a strategy, a justification or a solution to a problem. Constructing consists of assembling and integrating previous constructs by vertical mathematization to produce a new construct.

Some studies have used the AiC framework for investigate the processes of constructing knowledge in technological contexts (Anabousy & Tabach, 2015; Kidron & Dreyfus, 2010; Ofri & Tabach,

2013). These studies demonstrated the positive influence of technological tools. Specifically, Kidron and Dreyfus (2010) studied how instrumentation led to constructing actions and how the roles of the learner and a computer algebra system (CAS) become intertwined during the process of constructing a justification. They showed that certain patterns of epistemic actions were facilitated by the CAS context. Ofri and Tabach (2013) studied knowledge construction among eighth-grade dyads in a GeoGebra environment to explore a problem situation related to functions. They found that the students constructed the targeted knowledge while interacting with a dynamic and multi-representation environment.

The present study aims at tracing processes of constructing abstract mathematical knowledge among three pairs of seventh-grade students engaged in an inquiry task to construct the relations between areas of squares built on the sides of an obtuse/acute triangle in a GeoGebra environment. Also, the present study describes how the technological tool influenced the participants' actions.

Use of dynamic technology in mathematics education: the case of GeoGebra

Numerous studies have shown that dynamic technologies can be used to encourage exploration, conjecture, construction and explanation of geometrical relationships (Jones, 2005). Studies have also shown that the visual characteristics of these technologies may develop the ability to make correct assumptions (Hohenwarter et al., 2008). One such dynamic mathematical software system is GeoGebra, which is specifically designed for learning and teaching mathematics.

The educational potential of GeoGebra has been demonstrated by various studies that examined its effect on learning mathematics (see, for example, Dikovic, 2009). This tool has the potential to encourage student-centred and discovery learning by using interactive explorations to experiment with mathematical ideas (Tran et al., 2014).

As mentioned, the integration of GeoGebra software into teaching and learning mathematics has various benefits (Dikovic 2009; Erkek & Işıksal-Bostan, 2015). On the other hand, this integration also has its disadvantages (Jones, 2005; Erkek & Işıksal-Bostan, 2015). Research on the effectiveness of integrating GeoGebra in teaching and learning mathematics is still limited (Dikovic 2009; Saha et al. 2010). In the present study, GeoGebra was used to build carefully designed applet used in an inquiry task to examine the process of mathematical knowledge construction in this context.

The study was guided by the following research questions:

1. How do seventh-grade students construct the expansion of the Pythagorean Theorem: the case of changing the right-angle triangle to obtuse/acute triangle?
2. How do the purposefully designed GeoGebra-applet influence the construction process?

METHOD

Three pairs of seventh-grade students from the same class participated in the study. According to their teacher, all had high mathematical achievements.

An appropriate GeoGebra applet and an inquiry task concerning the relations between areas of squares built on the sides of an obtuse/acute triangle were designed for the study.

The task presented a mathematical situation (Figure 1). The students were asked to propose a hypothesis regarding the mathematical situation and then to experiment with GeoGebra to verify or refute their hypothesis. Finally, they were asked to explain/justify the constructed mathematical

concept/relation. The students worked on the task for about 45-55 minutes, and their work was recorded and transcribed verbatim.

Do you think there are relations between areas of squares built on the sides of an obtuse/acute triangle? Explain!

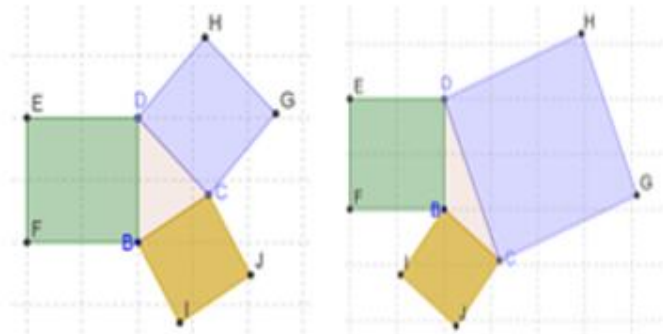


Figure 1: The mathematical situation presented in the task

GeoGebra was selected as the technological environment due to its dynamic nature and ease of use. One GeoGebra applet was built for the study by the first researcher. Figure 2 provides a screenshot of the interface of this applet.

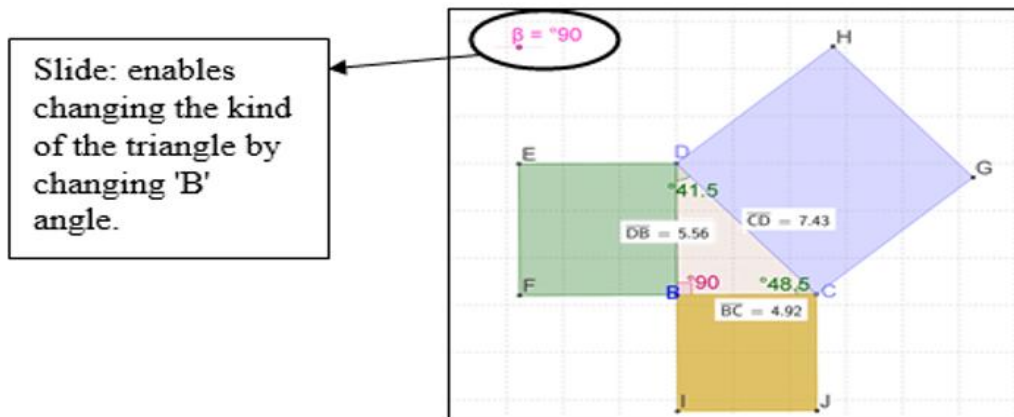


Figure 2: screenshot of the interface of this applet

The expected knowledge elements and sub-elements to be constructed were assumed based upon a-priori analysis. We also considered the Pythagorean Theorem to be a previous knowledge element because of its critical role in the construction processes we assumed would occur. The Pythagorean Theorem was constructed by the three pairs in a task designed by the researchers in a previous study (Anabousy & Tabach, 2015).

Figure 3 shows the a-priori analysis of the connections between the knowledge elements subsequently described. An operational definition was developed for each element to guide the analysis of the students' knowledge constructing activity.

E1: The relations between areas of squares built on the edges of an obtuse triangle.

E2: The relations between areas of squares built on the edges of an acute triangle.

E3: The justification of E1.

E4: The justification of E2.

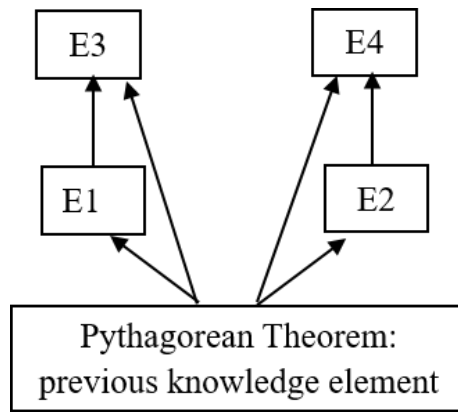


Figure 3: The connections between assumed knowledge elements

Here we consider that expected knowledge elements to be constructed as (E_i), while when the student constructs that element we refer to this process as construction (C_i) of the element.

FINDINGS

We describe the process of constructing knowledge graphically and verbally. We also present an episode that shows the constructing of knowledge in the technological context.

To describe the knowledge construction process graphically, we chose the following representations which include all possible cases. When a knowledge element is in the process of being constructed, a solid rectangle appears under it (■). When there are no acts of construction, the rectangle is empty (□). When the construction is successfully completed, a bold black line (▬) appears at the bottom of the rectangle. When the construction ends partially, a bold red line (▬) appears at the bottom of the rectangle. When an unexpected element is constructed, the rectangle is dashed (▭). When a knowledge element is constructed incorrectly, the rectangle is red (■). Finally, when a knowledge element has not been constructed at all, a solid rectangle framed in red appears (▭).

We divided the activity into ten segments that are parallel to the activity questions. These segments are arranged chronologically: (1) Recording the areas of squares built on the sides of an obtuse-angled triangle; (2) Identification of the relations between the areas of the squares built on the sides of an obtuse-angled triangle and its generalization; (3) Explanation of the relations between the areas of the squares built on the sides of an obtuse-angled triangle; (4) Problem formulation for exploring the case of the acute-angled triangle; (5) Exploration of one case (of the acute-angled triangle) and generalization; (6) Adjustment of generalization; (7) Checking the students' confidence in their conclusion; (7*) Discovery of another connection (unexpected knowledge element constructed by the first pair); (8) Last formulation of the conclusion; (9) Explanation of the connection between areas of squares built on the sides of an acute-angled triangle. Figure 4 below describes the process of knowledge construction by the three pairs in the activity graphically. In the following figure 4, the vertical axis represents the ten segments, while the horizontal axis represents the knowledge constructs.

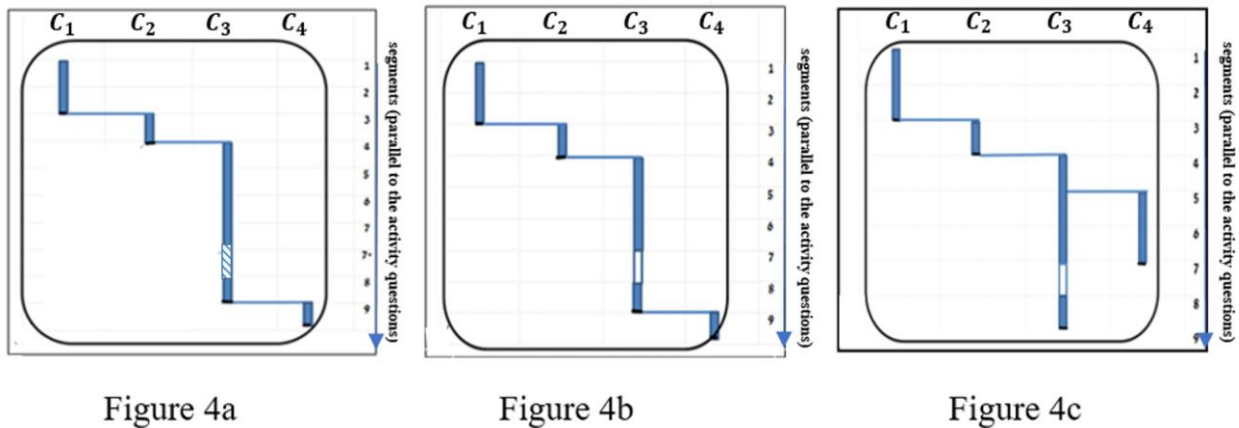


Figure 4: Knowledge construction in the activity: first, second and third pairs, respectively.

Figure 4 shows the order of construction. As we assumed, the constructs C1, C2, C3, and C4 (usually) were successfully built one after the other by the three pairs. The figure also demonstrates that the first pair produced an unexpected construction: as one of the angles of the triangle approaches 90° , the sum of the areas of the two squares built on both sides of the angle approaches the area of the square built on the opposite side (this process is described in episode 1). Based on Figures 4a-4c, we can also claim that by and large, all the pairs underwent similar construction processes. The process of constructing C4 was similar for the first and the second pairs, and different for the third pair, with the difference manifested in the time it took to build the construct.

Below is episode 1, which shows construction of the unexpected construct by the first pair. Here, R indicates *recognizing* action, B indicates *building-with* action and C indicates *constructing* action.

Episode 1: Construction of the unexpected knowledge element (S1 and S2 are the students)

1. S1 This is a right angle triangle [presented in fig.5].
2. B_Pythagorean theorem S2 No, we cannot depend on the picture. $22+9=27$... it's 'more than' relation [she means: the sum of the areas of the two squares built on the edges comprising the acute angle is more than the area of the square built on the edge opposite to the acute angle]. $9+13=22$...ohhh what?!!
3. S1 Something unusual has happened!
4. Inter. What?
5. S1 S2, You were wrong, this is not 13, it is 14 if we approximate it to an integer [means to approximate 13.8 to integer number]
6. B S2 Yes, $14+9=23$, it's a 'more than' relation.
7. Inter. You do not have to approximate the numbers. Look at this triangle; it is close to being what?

- | | | | |
|-----|------------------------|--------|--|
| 8. | R_right-angle triangle | S2 | A right-angle triangle. |
| 9. | | Inter. | And what happens to the areas? |
| 10. | B
C | S2 | Ahhh... the sum of the areas of the two squares built on both sides of the angle, the angle that is approaching 90° , approaches the area of the square built on the opposite side. |
| 11. | | S1 | That's it, we're finished. |

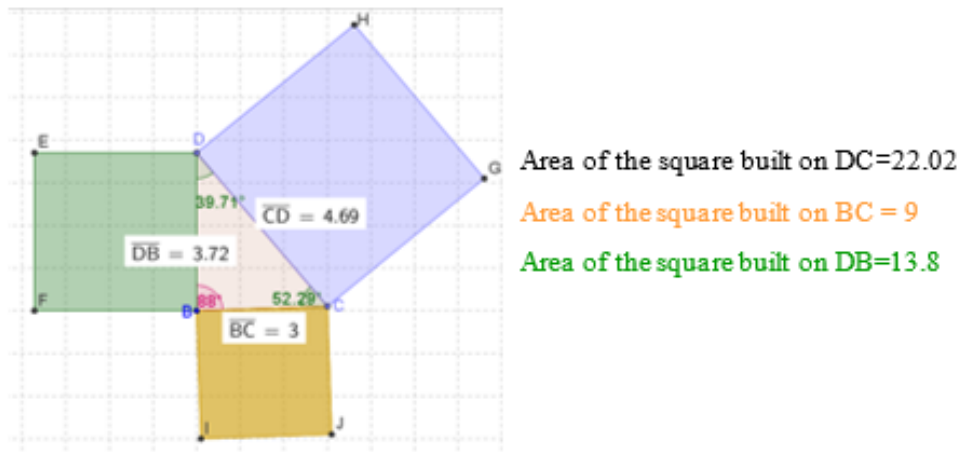


Figure 5: Screen shot from the work of the second pair in episode 1

In this episode, the students investigated an extreme case (see Fig. 5). This investigation enabled the students to move from constructing E2 to constructing the unexpected construct in line 10. The students watched the change in the areas of the squares built on the sides of an acute triangle, and formulated two knowledge constructs: (a) as one of the angles of the triangle approaches 90° , the sum of the areas of the two squares built on both sides of the angle approaches the area of the square built on the opposite side [line 10]; and (b) E2 [in discussion after episode 1, not shown].

As mentioned, the knowledge construction processes of these students occurred in a technological context. The technological tool (GeoGebra applet) supported the processes of constructing knowledge, as it enabled students to (1) explore "representative" cases such as triangles with different "types" of side lengths: large/small numbers, fractions and integers and specific extreme cases (e.g., when constructing E1 and E3); (2) transition from one construct to two parallel constructs and constructing an unexpected construct. This transition took place during the construction of E2 by the first pair (see episode 1, lines 1-3); (3) justify/explaining the constructed relations (when constructing E2 and E4). Below we present in details the process of constructing the explanation of E1 and E3 by the first pair.

Constructing C_2 and C_4 (explanation of C_1 and C_3) by the first pair (S1 and S2 are the students): S1 performed the construction immediately at the beginning of the question, with the help of knowledge elements C_1 and C_3 . She said, "...in an acute-angled triangle, the right angle decreased [refers to the method they use to obtain an acute-angled triangle]; therefore, the side was reduced, and therefore the area of the square was reduced. And they [squares] were bigger than it, as opposed to the obtuse." S1 tried to explain this to S2, but S2 was not convinced. Eventually, S1 suggested to

S2 to use her hands and began to explain to her what was happening in an acute-angled triangle and in an obtuse-angled triangle, while examining the case of a right-angled triangle. Figure 6 shows an example of S1's explanation of this connection.

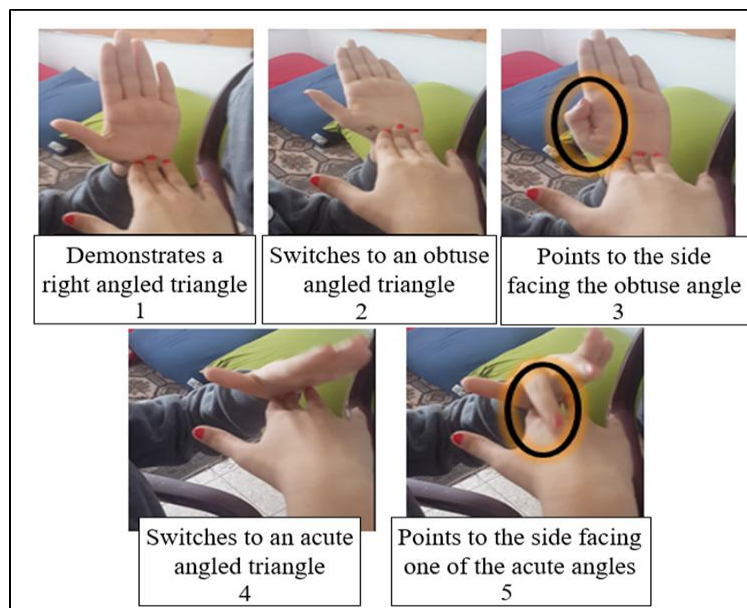


Figure 6: Gestures used by the first pair while explaining the expanded Pythagorean Theorem

DISCUSSION

The present study traced processes of constructing mathematical knowledge by three pairs of seventh-grade students engaging in an inquiry task to construct the relations between areas of squares built on the sides of an obtuse/acute triangle in GeoGebra environment. The findings indicate that the three pairs constructed all the knowledge elements. Successful construction of knowledge in a technological context has also been reported by Ofri and Tabach (2013).

GeoGebra applets supported the knowledge construction sequence: examination of various cases, generalization, proving or explaining. For example, when constructing C_1 (The relations between areas of squares built on the edges of an obtuse triangle), pairs of students discovered the relation by considering many different cases, which was made possible by manipulating the applet. The various cases were not random, as the students selected specific representative cases, such as polygons with sides measured in integers, fractions, large and small numbers. This examination allowed students to make a generalization regarding the relation. Dikovic (2009) also reported on support provided by the technological tool GeoGebra in exploration and generalization activities.

The technological tool also allowed two parallel construct actions to take place simultaneously and it enabled unexpected constructs to be built. For example, when the first pair was constructing C_3 (the relation between the areas of squares built on the sides of an acute-angled triangle), the students observed the change in the areas of the squares built on the sides of an acute-angled, they also examined extreme cases which were provided by the applet and then built unexpectedly two knowledge constructs (for similar findings see Kidron & Dreyfus, 2010)

The construction process was further supported by the technological tool by reducing the need to spend time on calculations, allowing the pairs to focus on searching relations and explanations, for

example when constructing C_1 and C_3 . The tool provided the areas of the shapes. Many studies have reported similar contributions of the technological tool, for example Becta (2003).

The technological tool also supported the explanations given by the pairs, for example the explanation of C_2 . All the pairs argued that when transitioning from a situation in which the triangle is right-angled to one in which the triangle is obtuse-angled, the side opposite the angle will be longer, so that the area of the square built on it will be larger, which will change the relation from equivalence to one of "bigger than." In this case, the structure of the applet and the work using it supported and facilitated the emergence of the explanation provided by the students (Lachmy & Koichu, 2014). Several studies (Ng & Sinclair, 2013) showed students' reliance on gestures and dragging to be multimodal resources for communicating about dynamic aspects of mathematics.

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