GEOGEBRA AND NUMERICAL REPRESENTATIONS: A PROPOSAL INVOLVING FUNDAMENTAL THEOREM OF ARITHMETIC

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This paper reports a qualitative research whose subjects were Elementary School Teachers who took part in a workshop about primality of positive integers and the Fundamental Theorem of Arithmetic (FTA). These topics was dealt with from different technological perspectives and analysed under a theoretical proposal connected to the concepts of transparency and opacity of numerical representations and to the "humans-with-media" approach. The interactions occurred in a Post Graduate Program in Mathematics Education and they consisted of two activities created to ask subjects which numbers in a random list would be prime. The analysis showed that participants had difficulties with FTA, which led them to adopt strategies with a high cognitive cost and make mistakes. Likewise, data showed that hindrances were overcome based on the educational proposal planned from a configuration of humans-with-GeoGebra.

Keywords: fundamental theorem of arithmetic; numerical representations; digital technologies in education; human-with-media; GeoGebra.

INTRODUCTION

The task of recognizing whether a positive integer is prime may seem simple and not very interesting in terms of teaching or researching in the field of Mathematics Education. Considering that a natural number is prime if it can be divided uniquely by itself and by one, we might think that there are no associated complexities. Ultimately, it just seems to be a question of finding a divisor between 2 and the square root of the number tested for primality or of applying the so-called ‘divisibility rules’. Such a search, however, can have quite a high operational cost. For example, if the number is 30847, we must consider that its first divisor is 109 and that there is only another divisor apart from the number itself and 1, which is 283.

In order to deal with such problem, this work describes a research whose participants were a group of basic education teachers from public schools, involved in the projects “Mathematics Teaching and Learning Processes in Technological Environments”, constituted through a partnership between the Pontifical Catholic Universities (PUC) of Sao Paulo (Brazil) and of Lima (Peru), and “Technologies and Mathematical Education: researches on fluency in devices, tools, artefacts and interfaces”, also linked to the PUC Sao Paulo [1]. In the proposed activities, teachers had to identify whether several numbers were prime, in situations where divisibility rules, for example, were an inefficient strategy and where FTA knowledge would be relevant. Through the chosen research instruments – a question about the subject ‘primality of natural numbers’ and an application built with GeoGebra – we sought to highlight the strategies used by the subjects to solve the problems, their ideas about the representation of natural numbers and the influence of the type of technology on the mobilization of the mathematical knowledge at issue. In this respect, arguments related to the number representation arise, as well as discourses related to the use of technology in Mathematics Education, which leads to the following theoretical treatment.

NUMERICAL REPRESENTATIONS

One of the core concepts addressed in this research refers to transparency and opacity of numerical representations. In this respect, the study of Zazkis and Liljedahl (2004) mentions the role of representations in the field of natural numbers. In their paper, the authors discussed data obtained from a research conducted with basic education teachers in training, which focused on subjects understanding of prime numbers, so as to detect the factors that interfere with such understanding. The argument used in the analysis of collected data is that the lack of transparency of prime numbers representation is a hindrance to understand them. This idea was inspired from the paper of Lesh, Behr and Post (1987). When referring to different representations of rational numbers, the authors show that they “incorporate” mathematical structures, meaning that they represent them materially. Thus, representational systems can be regarded as opaque or transparent: a transparent representation has no more or less meaning than the ideas or structures it represents, while an opaque representation emphasizes some aspects of the ideas or structures it represents while hiding others. When having different representational possibilities, a didactic strategy should, for example, capitalise on the strengths of a specific representational system and minimize its weaknesses.

Expanding Lesh, Behr and Post’s proposal (1987), Zazkis and Gadowsky (2001) introduce the notion of relative transparency and opaqueness, focusing on numerical representations. The authors suggest, on their paper, “that all representations of numbers are opaque in the sense that they always hide some of the features of a number, although they might reveal other, with respect to which they would be ‘transparent’” (p.45). As an example, the authors provide a list with the following items: (a) 2162, (b) 363, (c) 3 x 15552, (d) 5 x 7 x 31 x 43 + 1, (e) 12 x 3000 + 12 x 888. The authors mention that such expressions do not seem to represent the same number, 46656, pointing that each representation shifts our attention to different properties of number.

Based on the above-mentioned concepts, the authors claim that all numeric representations are opaque, but they have transparent features. In relation to the work we describe here, activities involving the features of numerical representations were presented to the subjects by using different means and technological interfaces. For this reason, a discussion about the use of technologies in Mathematics Education becomes necessary in this paper.

ABOUT THE USE OF TECHNOLOGIES IN MATHEMATICS EDUCATION

Technologies have been part of educational processes from time immemorial, if we consider, like Lévy (1993) that transmutation of temporalities in Human History brought about different instruments; some of them prevailed over the others depending on the evolutionary character of the society in a given time. Thus, orality, writing and information technology are enrolled as technologies of intelligence. The ascendancy of information technology did not suppress previous technological proposals, but constituted, in relation to them, a feature of redefining functions and, in the last resort, of convergence. Thus, it is unavoidable to associate the use of some technology in processes of teaching or learning: a construction of knowledge and its forms of access, therefore, were always linked to more or less material tools of technological nature.

From this perspective, we can conjecture that the process of knowledge construction involves people and technologies associated in some way. However, technologies are not part of this association as substitutes for human capacities, not even as a supplement to them, but as reorganizers of human thought (Tikhomirov, 1981). In this sense, computer applications, for example, allow for unusual forms of mediation, delegating to the computer the role of tool for human mental activity, with functions similar to those performed by language in vygotskinian logics. Similar reasonings apply to contemporary devices such as tablets or cell phones.

Such considerations support the claim that, in Mathematics, learning is a process involving technologies that are somehow integrated with people what allows intentions, strategies, plans and conjectures come into play. According to Borba and Villarreal (2005), such integration must be of such an order that it excludes any attempt to see these items, people and technologies, as separate groups. Therefore, for these authors, mathematical knowledge is formed from a collective of humans-with-media, considering that media reorganize people's thinking and that the presence of different technologies conditions the production of different forms of knowledge. Thus, in the research described here, the two activities explained below attempted to investigate how teachers in continuing education comprise numerical representations related to prime numbers and FTA, based on the use of different technologies at different times: during the first activity, teachers-with-pencil-and-paper; in the second activity, teachers-with-computers-and-GeoGebra.

METHODOLOGICAL APPROACH

The participants of this qualitative research are eight basic school teachers in São Paulo and Pará states, all of them voluntary in workshops carried out in the framework of the projects mentioned at the start of this text. The research we describe here was conducted in one of the computing labs of the abovementioned institution, in a single session which lasted around four hours. Among the teachers so described, five work in primary and secondary education while three of them work only in primary education. Moreover, all of them have completed their bachelor's degree in Mathematics, three of them are attending an Academic Master in Mathematics Education and two have completed their Specialization in Mathematics Education.

Participants in this study were invited to working with two kinds of activities involving knowledge on primality in the framework of the Theory of Numbers. The first activity consisted of an issue to be solved individually (here, the response had to be presented in writing): "Consider F = 151 x 157. Is F a prime number? Circle YES/NO, and explain your decision" (Zazkis & Liljedahl, 2004, p.169). To answer to the abovementioned question, students would have to write down the option 'No', since the representation in question, with transparent features, shows that F is composite, and it even relates the component prime factors. Teachers should resort to the Fundamental Theorem of Arithmetic (FTA) to conclude that the said decomposition would be unique, except by the order.

If they considered another strategy, subjects could perform the multiplication contained in the question, obtaining 23707, an opaque representation in relation to the detection of the composite character of the number. From this other representation, teachers could test with several possible divisors, although this was not necessary to solve the problem. It might even happen that some teacher attempted to divide 23707 by the prime numbers from 3 onwards, giving up after some attempts, since natural numbers below 151 are not divisors of 23707 – in this case, this teacher could even erroneously state that the number would be prime.

The next activity was carried out immediately after the first one. Individually, the eight teachers had in front of them a GeoGebra screen containing only one button with the word "Números" ("Numbers") on it. They were all informed that the application would draw nine numbers when they pressed the said button. Then, researcher instructed teachers with regard to the result they would see at clicking the button: nine numbers would be shown on the Algebra View of the software. They had to state, by writing on a blank paper that had been handed out and in a lapse of 20 minutes, whether each drawn number was prime or not. In the meantime, they should not click on the button with label “Coisa” (“Do something”) [2] on it (Figure 1) shown after the draw.

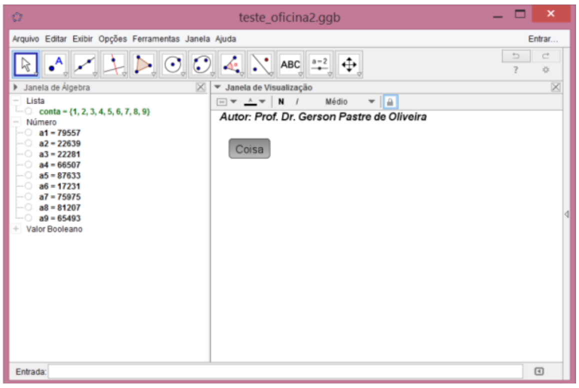

Figure 1. Screen with drawn numbers (developed by the author)

Regarding the numbers, it is worth noting that the construction of the javascript code that carried out the draw took into consideration the random selection of numbers in the range of 1001 to 99999 (Figure 1). The purpose was to restrict the direct application of rules and division algorithms. Despite this, these resources could be successfully used in some cases (such as 47301, which is divisible by 3) and fail in others (such as 39203, which is not prime, but whose prime factors are 197 and 199). In theory, the representations provided by the software were opaque with respect to the primality of the numbers, at least up to this point of the experiment.

Once the time allowed was over, regardless of the amount of resolutions among the nine proposals, subjects were invited to click on button “Coisa”. After this action, the Algebra View of GeoGebra showed the decomposition of each of the nine numbers into prime factors, in the form of lists (Figure 2) – in the case of primes just the number itself was presented.

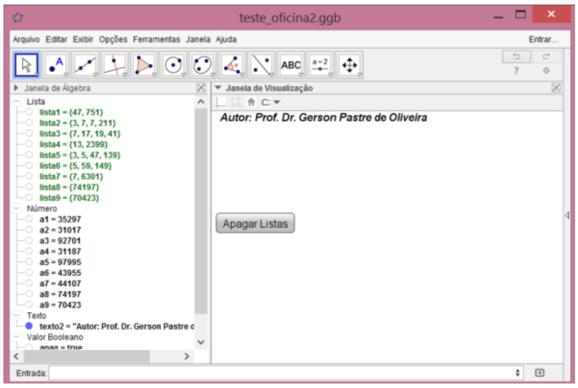

Figure 2. Numbers and their compositions in prime factors (developed by the author)

Immediately after this action and still individually, participants were invited to revisit their previous answers in the light of the new data obtained with the factorization carried out in GeoGebra. At that occasion, subjects were expected to relate the lists obtained with the numbers drawn previously and use the FTA so as to decide on the primality or not of each one. Ten minutes were allowed for this part of the activity, after which there would be a debate involving the subjects and the researcher. The button “Apagar Listas” (“Erase all”) (Figure 2) could be used to go back to the initial screen, which would allow repeating the experience as many times as considered necessary by participants.

ANALYSIS

At the very beginning of first activity, Teach2 after trying some division operations with pencil and paper stated that 23707 would 'probably' be prime. Other four participants also stated, wrongly, that the number in question would be prime, using similar strategies:

- Teach3 did countless division operations and ended up stating that "23707 is prime, because it can be divided by itself and by one". In this way, Teach3 showed he is not aware that this criterion does not distinguish between prime and composite numbers;
- Teach4, after several attempts using division operations, concluded that 23707 would be a prime number because it "ends in 7 and 7 is prime";
- For Teach5, as the divisibility tests by 2, 3, 5, 7, 11 and 13 'failed' (divisions had a remainder other than zero), the number in question would be prime – in this case the teacher indicates he believes that "decomposition into prime factors means decomposition into small prime factors" (Zazkis & Campbell, 1996, p. 215);
- In the case of Teach8, number 23707 "is prime, because it is an odd number and it cannot be divided by its square root or any other prime number". Like Teach5, Teach8 limited the universe of prime numbers to the interval between 2 and 13 and it evidenced several confusions involving the concepts of perfect square numbers and odd numbers.

The representation of F provided in the question formulation had transparent features in relation to primality, because it presented the number by means of its unique decomposition in prime factors. However, the abovementioned teachers did not use this idea as expressed in the FTA, which points out the fact that a numeric representation with features that make it transparent can be kept opaque when mathematical knowledge about it is not mobilized by the individual. Similar results were observed in the work of Zazkis and Liljedahl (2004), which leads us to consider the need to use didactic strategies capable of reinforcing the transparent characteristics of this representational system, as advocated by Zazkis and Gadowvsky (2001). This was done in the second activity.

Furthermore, some participants correctly stated that F would not be prime. According to Teach1, "F is divisible by 151 and 157, which makes it non-prime". Participants Teach6 and Teach7 said, in a similar way, that F had other divisors besides itself and 1, which would make it non-prime. Nevertheless, none of the three participants that answered correctly did show any sign of using the FTA in their conjectures: when questioned about the possibility of F having other divisors apart from the ones mentioned, all of them said it was possible but that they would have to test numbers up to a certain limit (according to Teach1 up to the number square root; according to Teach6 and Teach7, up to half the number). Another aspect worth highlighting is that none of the teachers showed awareness of the fact that factors 151 and 157 represented prime numbers.

From a different perspective, the technological component of the collective humans-with-pen-and-paper, even if intensively used, does not seem to support cognitive movements related to a change of strategies in this activity – in the case of participants who exhausted attempts with divisibility algorithms – or to the use of formal notions in Mathematics, such as the FTA – in the case of participants who stated that 23707 might have other divisors.

Regarding the second activity, carried out in GeoGebra, participants accessed the application that was available on the computers in the institution's lab and drew the nine numbers, as shown on figure 1, without further questions. From this moment, teachers had 20 minutes to decide which numbers would be primes. All participants alleged, initially, that time was too tight and that numbers would be too big (odd numbers between 1001 and 99999) to be able to provide an answer. The researcher said that they should state as many results as possible in this lapse of time, which could not be extended.

Until that moment, the technological aspect in the collective teachers-with-GeoGebra did not have great influence on the issue of transparency of the numeric representation regarding primality, because the program interface in question only provided odd random numbers within the said limits. Thus, as drawn numbers were potentially different, the amount of correct or wrong results showed significant differences among the subjects. Those whose drawn numbers allowed direct application of divisibility rules or tests with 'small' prime divisors (3 to 13) obtained more hits than those whose drawn numbers were 11741 (59 x 199) or 31753 (113 x 281), for example. Generally, such numbers were wrongly stated to be prime. Even when prime numbers were identified, as 17231, a doubt used to remain, as in the case of Teach5, who wrote, next to the said number, "I think it is prime. I tested up to 13". As we have already seen, this strategy, in the case of numbers whose prime factors are all higher than 13, is not efficient.

Among the teachers' talks during the 20 minutes allowed for the activity, several references were made to the numbers 'difficult form', a clear attempt to refer to their representation, which is clearly opaque in terms of the feature 'primality'. Teach6 even got to question the goal of GeoGebra in that context, as the application did not seem, according to him, "to make things easier for those who tried to solve the problem".

Faced with perplexity caused by the proposal, the researcher, after the time allowed, proceeded to coordinating a debate with the participants, whose main motivation was raising the conjectures and strategies that subjects had proposed in order to verify the primality of their nine numbers. None of the participants said to have tried or even thought about obtaining the factorization of the numbers to be tested for primality in order to using knowledge on FTA. Once the debate was ended, the researcher said that subjects could click on button 'Coisa', which would show, for each raffled number, the respective list of component prime factors (figure 2). After this, in 10 minutes, teachers had to review their answers. Immediately after clicking on the button, teachers had to construe the data showed on the screen.

When they realized the correspondence between the lists provided and drawn numbers (list1 referred to number a1, list2 referred to number a2, and so on), most teachers started searching for relations among the said components:

Teach6: Professor, I would like to review my answers.

Researcher: Yes, why so?

Teach6: Because I realized something that I had not seen before… the lists… they are the factors of each number...

Teach1: By multiplying the numbers in the lists we obtain the drawn numbers…

Teach4: It´s true, but there are cases where only one number appears… these numbers are primes, as we can only multiply by one!

Teach3: [does not seeming convinced] Professor, I´ll draw the numbers again… [after repeating the draw and factorization] It's true! Prime numbers do not have factors, only composite numbers do.

Teach4: A prime number's factors are itself and one...

Teach7: I was thinking... In my case, one of the numbers is 88739... Factorization is shown as 7, 7 and 1811. I could as well write 49 and 1811, right?

Teach8: [after some discussion with pairs] I think that 49 can appear, but 49 is not prime, and lists show prime factors of numbers. The idea is that only prime numbers appear. We can see that all the factors, in all the numbers, are prime.

Teach4: You are right. Any number can be written as a product of prime factors! This is it! Wow! First question was obvious! There is only one decomposition into prime numbers for each number.

Teach8: It is the Fundamental Theorem of Arithmetic!

After these observations, subjects could tell which numbers were prime and which were not, by repeating raffling and the whole procedure several times. As noted by Borba and Villarreal (2005), visualization and experimentation were important factors in the new strategy adopted by subjects from configuration humans-with-GeoGebra. According to these authors, such items allow, among other actions, for example, to invest in generating conjectures about the problems at issue (and testing them through countless examples), bring to light some results which were not known before the experiments and testing different ways of collecting results. The access to visual components, in the consolidation of the results of actions carried out by people-with-GeoGebra, became a way to transform the understanding they had about the problems at issue.

Another aspect that must be taken into account in the configuration teachers-with-GeoGebra is the dynamism of digital technologies that has been seen here as a possibility to manipulate the parameters, attributes or values which served the constitution and/or definition of a mathematical construct in a computerized context. Faced the possibilities open by this resource, a fundamental investigative movement to mathematics finds consistent subsidies in the development, testing and validation (or refutation) of conjectures. This could be largely seen in the experiment we describe here, when teachers invested, through experimenting and visualizing, in the procedure repetition, using the regularities observed in factorizations – and dynamically obtained - as a means to support the reorganization of ideas on the primality of the presented numbers. All these factors collaborated to mobilise knowledge on FTA for the solving of the problem.

FINAL CONSIDERATIONS

The discussion following the last activity was quite fruitful: participants stated that, in activity 1 they had not realized that the 'way' in which the number was written (its representation) allowed the question to be answered directly by mobilizing knowledge related to the FTA.

Regarding activity 2, participants said that numbers were not expressed in a 'convenient way' (transparent representation), i.e., according to them, they were 'big numbers' that were not decomposed in prime factors. Teachers mentioned the fact that they had spent the 20 minutes to try and say which of the numbers were prime, but if they had the appropriate representation in factors and had remembered the FTA, they would have done this much faster. This last feature was perceived when they clicked on the second button, causing the decomposition of the numbers in prime factors to appear. The participants concluded that, when there were other factors besides the number itself, the number in question would not be prime. Moreover, teachers highlighted the importance of FTA knowledge and of the use of GeoGebra in the procedures, saying that this would be a good way to approach the topic in classroom.

Moreover, in this context, the Geogebra´s application would work as a 'calculator to decompose in prime factors', turning a numeric representation, opaque in relation to primality, into a transparent representation, provided that the knowledge about the FTA has been mobilized. In this case, the configuration teachers-with-GeoGebra contributed in a more efficient way to direct the problem-solving effort towards a strategy that brings more chances of success. Dialogues show, although only to some extent, the renegotiation of meanings, the conjectural reformulations and the reorganization of thinking which allowed the right answer to come out. Another feature that must be highlighted is that representations of prime numbers can give opportunity to transparent features emerge, as soon as the appreciation of underlying meanings and concepts is taken into account. When this does not occur, the tendency is to call for large, expensive solutions in cognitive terms. To investigate the nature of these processes and develop proposals in order to avoid such difficulties to remain among basic school teachers seems to be an important challenge, open to new researches.

NOTES

1. Research projects supported by FAPESP and CNPq, respectively (Brazilian scientific development agencies).

2. The button title could then be 'Factorize', but the idea was to promote a research where subjects were the authors of the hypothesis formulated to solve the problem: the use of this title might imply that teachers needed, compulsorily, to do the factorization of the numbers, which would compromise their autonomy.

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