"POWER OF SPEED" OR "DISCOVERY BY SLOWNESS": TECHNOLOGY-ASSISTED GUIDED DISCOVERY TO INVESTIGATE THE ROLE OF PARAMETERS IN QUADRATIC FUNCTIONS

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This paper reports an intervention-study where students investigated the role of parameters in quadratic functions through technology-assisted guided discovery. The intervention had three experimental groups of students, each of which used a different type of visualisation *(sliders, "drag mode", or a function plotter) and one control group. The study provides insight into whether technology-assisted discovery learning supports the conceptualization and understanding of the role of parameters in quadratic functions. Qualitative analysis of students' work investigated the potential and constraints of each of the four different approaches for visualisation. Initial findings showed that technology does support the students in their learning, with the dynamic visualisation groups (drag mode and sliders) showing greater understanding than the function plotter group.*

Keywords: quadratic functions, dynamic visualisations, parameter, discovery learning, technology

BACKGROUND

During the study, the students take part in a self-paced guided discovery learning about the concept of parameters in the field of quadratic functions. Mosston and Ashworth (2008) describe guided discovery as a "convergent process that leads the learner to discover a predetermined target" (p. 214) whereas Gerver and Sgroi (2003) stress the importance that guided discovery lessons "have a story line that inherently engages the participant" (p. 6). The lessons also need an "Aha! Component" (Gerver & Sgroi, 2003), so at some point in the lessons the students realize that they discovered mathematical ideas through their exploration. Even though minimal guidance is widely promoted, Kirschner, Sweller & Clark (2006) state that there is no research supporting the use of instruction using minimal guidance. However, Alfieri, Brooks, Aldrich, and Tenenbaum (2011) in a meta study of 164 studies of discovery learning found that enhanced discovery learning, for example through use of guided tasks, was indeed beneficial for learning.

In order for students to develop conceptual understanding of parameters it is important to have an understanding of both variables and functions, as well as being able to change between different representations.

Variables can be used differently in different contexts. For example, a variable can be used as a placeholder or as an unknown, depending on the context of a problem. Küchemann (1981) described six ways children use letters in mathematical tasks: *letter evaluated, not used, used as an object, used as a specific unknown, used as a generalised number* and *used as a variable*. Usiskin (1988) however, describes the variable as a "symbol for an element of a replacement set" (p. 9) but also states that there are different views possible. He distinguishes variables as *pattern generalizers, unknowns, parameters* and *arbitrary marks* on paper (Usiskin, 1988). Küchemann and Usiskin use different terms for the same concept, for example Küchemanns' *letter evaluated* and Usiskin's *unknown* both describe the same use of a variable. In contrast, Malle (1993) only describes three roles of variables namely, variables as *unknowns, generalized number* and *changing variable*. Some authors, for example Usiskin (1988), view parameters as a specific role of variables, whereas others, for example Drijvers (2003), describe parameters as meta-variables with several meanings themselves. Drijvers distinguishes between parameter as a *placeholder*, as a *generalizer*, as a *changing quantity* and as an *unknown*. The view of a parameter as a *placeholder* can result in consideration of specific values, one by one, for the parameter; in a graphic model, each time the parameter is changed it is visualized as one graph being replaced by another. However, graphs can also be viewed dynamically, and therefore the parameter as a *changing quantity* can be observed via dynamic software (e.g. graphics calculators, or computer software) while the parameter is continuously changed to take a number of values. Lastly, a parameter as a *generalizer* can be visualized by a family of functions (Drijvers, 2003). Drijvers (2003) suggests a hierarchy for the understanding of parameters, with parameter as a placeholder associated with a lower level of understanding than both parameter as a changing quantity and an unknown, while understanding of a parameter as a generalizer shows a higher level of understanding. Students need to be able to distinguish between parameters and variables, but this presents difficulties as the distinction is context related and parameters cannot be explained without second order structures (Bloedy-Vinner, 2001). Bardini, Radford, and Sabena (2005) describe this as "the paradoxical epistemic nature" (p. 130) of parameters, which makes it difficult for students to understand. Therefore, parameters need to be addressed in a variety of ways (Bardini et al., 2005).

In addition to understanding the concepts of variables, it is necessary for students to understand the Grundvorstellungen associated with functions. Grundvorstellungen and the development of Grundvorstellungen are terms used in the German literature to describe the connection between the mathematical concepts, real contexts and the students' mental models (Blum, 2004). Grundvorstellungen includes normative, descriptive and constructive aspects, where the normative aspects of Grundvorstellungen can be used to determine, what a full understanding of a particular mathematical concept should include. These normative aspects are derived through a subject matter analysis (vom Hofe & Blum, 2016). Descriptive aspects of Grundvorstellungen are used to describe the student's mental representation and these can include misconceptions or partial understandings (vom Hofe & Blum, 2016). Grundvorstellungen can be constructed through teaching, hence the constructive aspects of Grundvorstellungen. Blum (2004) identified the Grundvorstellungen of a function as mapping, covariation and object. Mapping is aligned with a static view, where one quantity is matched with another. Covariation, however is more aligned with a dynamic view, where a change in one quantity is observed when the other quantity changes and this dynamic view can be supported through the use of technology. Object as a whole is a global view of the function as one object (vom Hofe & Blum, 2016), which is different to the other two ideas of mapping and covariation, as the global features of the function are considered. vom Hofe, Kleine, Wartha, Blum, and Pekrun (2005) describe the linking of different Grundvorstellungen as an essential requirement for developing mathematical understanding. For the study presented here, it is therefore crucial that the three Grundvorstellungen of functions are developed, when students are learning about the differences between an original graph and the resultant graph as one of the parameters of a function equation changes.

As mathematical objects cannot be accessed without the use of representations, understanding the concept of a function is closely intertwined with being able to change from one representation to another (Duval, 2006). Duval (2006) describes being able to change between representations as a "critical threshold for progress in learning" (p. 107). He distinguishes two kinds of transformations of representations: treatments and conversions. Treatments occur within one register, for example solving an equation, whereas conversions transform one register into another, for example graphing a function from its equation (Duval, 2006). Using multiple representations have been found to be important in the teaching and learning of all mathematical concepts, with Kaput (1992) describing the change between different representations as a part of "true mathematical activity" (p. 524). Penglase and Arnold (1996) in their review of research on graphics calculators pointed out that this change between different representations can be supported through use of graphics calculators. Kaput (1992) described that the automatic linking possible through use of technology (so that a change in one representation immediately occurs in the other) can help students to visualize the connections between representations in a different way than a static change of representation. Some authors describe this linking of different representations as essential for developing a full understanding of concepts (e.g. Thomas, 2008; Duncan, 2010). This linking of representations can be supported through the use of digital technologies (Ferrara, Pratt & Robutti, 2005).

Using general-purpose technology like dynamic geometry software, spreadsheets or function plotters to explore mathematical connections is included in the core curricula in North Rhine Westfalia in Germany (MSW, 2007). It is not specified when or how it should be used, but starting from 2017 all students in North Rhine Westfalia need to use graphics calculator in their final upper secondary school exams ("Abitur"). Many studies have shown that dynamic technology can be beneficial for learning, however Zbiek, Heid, Blume, and Dick (2007) pose the question, whether the use of sliders obscures, rather than enhances, the understanding of a connection between the value of the parameter and the changing graph. Drijvers (2003) outlines that when students work with sliders, the students are often not successful in explaining the effects of the sliders as they only examine the effects superficially. However, Drijvers (2004) found that concerning the parameter as a changing quantity, students achieved a higher understanding through the use of sliders regarding his proposed hierarchy of parameter roles.

RESEARCH QUESTIONS

The main research question of this project is:

How can technology-assisted guided discovery support the conceptualization of parameters in the field of quadratic functions?

This paper reports on three sub-questions:

- Is it possible for a sample of 379 grade 9 students to support the learning of the concept of parameters in the field of quadratic functions through technology-assisted guided discovery?
- What insight into students' exploration and testing of hypotheses can be generated?
- What limitations of the study are visible?

METHODS, METHODOLOGY AND DESIGN OF THE STUDY

Participants

This study is a control group design intervention study with three experimental groups and one control group. 14 classes of grade 9 with a total of 379 students participated in the study. The 14 classes were from 8 different upper secondary schools from 5 different cities in two states in Germany (13 classes in North-Rhine Westphalia and 1 class in Thuringia). Ten teachers taught a single class and two teachers taught two classes each. In each school, where there was more than one class participating in the study, each class was assigned to a different group. A pen-and-paper technology-free pretest was conducted in all classes before the intervention to collect baseline data on pre-requisite algebra knowledge and skills for the intervention. This showed that the control group and experimental groups had similar skills in the field of linear functions and equations. Further information from the pretest is not reported in this paper.

Intervention: structure of the groups

All students took part in an intervention which involved the use of a given technology (using either a scientific calculator or TI-Nspire CX CAS) to investigate the role of parameters in the vertex form, $f(x) = a \cdot (x - b)^2 + c$, of a quadratic function. Table 1 provides an overview of the groups in the study, the type of visualisations and screenshots of pre-prepared files.

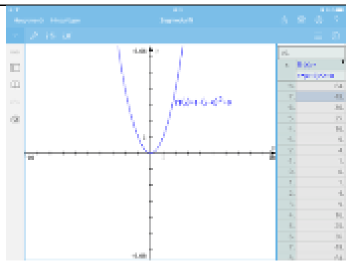
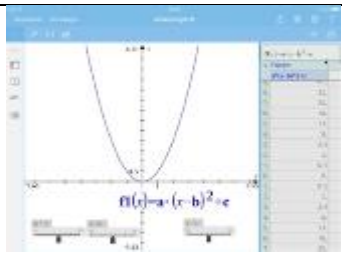
Group	Type of Visualisation: change of Parameters	Pre-prepared file provided to students
Control group "without visualisation" (n=71)	Function tables on scientific calculators, sketching graphs by hand	No pre-prepared file
Experimental group 1: "function plotter" (n=67)	Plotting functions and displaying function tables.	No pre-prepared file
Experimental group 2: "drag mode"(n=85)	Pre-prepared file, manipulation of the graphs by dragging the graph, dynamically linked function equation and tables	 **Figure 1: Screenshot of the "drag mode" file**
Experimental group 3: "sliders" (n=130)	Pre-prepared file, manipulation of the graphs through sliders for each parameter, dynamically linked function tables	 **Figure 2: Screenshot of the "sliders"-file**

Table 1: overview of the groups in the study

The **control group "without visualisation"** was only allowed to use a scientific calculator without graphing functions during the intervention, while the three experimental groups had access to the TI-Nspire CX CAS as handhelds or as an app on iPads and were able to use different features of the technology; namely, function plotters, drag mode or sliders.

- **"function plotter"-group**: students were allowed to use the "function plotter" freely and could choose how many and which functions to plot and produce function tables for.
- **"drag mode" group**: students were able to "drag" the graph and manipulate the form and position of the parabola. For each graph produced, the function equation and function table were displayed on screen and dynamically linked to the graph.
- **"sliders"-group**: students could change the parameters in the vertex form with sliders to manipulate the form and position of the parabola. For each graph produced, a general vertex form equation was displayed, without the values showing. The values for the parameters could be read from the sliders. The function table was shown and dynamically linked to the graph.

Intervention lessons:

The intervention consisted of three lessons of 45 minutes designed by the first and second named researchers. The first lesson involved an introduction to the use of technology features needed for lessons 2 & 3. This lesson was run by the teacher, or the first-named researcher where requested. This lesson is not reported here.

Lessons 2 & 3: Self-paced technology assisted guided discovery

Lessons 2 and 3 were conducted as a double lesson and were designed as a self-paced learning environment with guided discovery and the design process was empirically grounded. All students worked on the same worksheet which did not contain any instructions about technology use, but the students in the experimental groups were given a help-sheet for the use of TI-Nspire CX CAS System, where syntax for the most needed features was provided. Additionally, a pre-prepared file (refer to Table 1) was provided to each of the students in the "drag mode" and "sliders" groups. The main worksheet was structured in four parts, where students were asked to:

Part 1: Describe one example of a transformed parabola and explain the differences compared to a standard parabola.

Part 2: Explore the influence of the different parameters a, b, c in the vertex form

$$f(x) = a \cdot (x - b)^2 + c$$

one by one.

Part 3: Find explanations for the change in the graphs.

Part 4: Write a summary sheet in a form of a cheat sheet of everything learned during the lessons.

For parts 2 and 3 of the worksheet students were provided with a number of guiding questions on the worksheet. For example, in Part 2, the students were asked to first look at $f(x) = x^2 + c$ and change the value c.

Data

In 13 of the classes, one pair of students was videographed during the intervention and asked to think aloud and discuss their work with each other; these students (n=26) were told that it was not necessary to produce a summary sheet, however 22 students out of the 26 produced summary sheets. The videos were transcribed and qualitatively analysed. The first named researcher was present in these classes and wrote lesson observation notes. Summary sheets (for an example refer to Figure 3) were also collected from all other students (n=331), who produced the sheets in pairs or groups of three. In total, 178 summary sheets from 353 students were collected for analysis. The lesson observation notes, videos, video transcripts and the summary sheets provided data for this study.

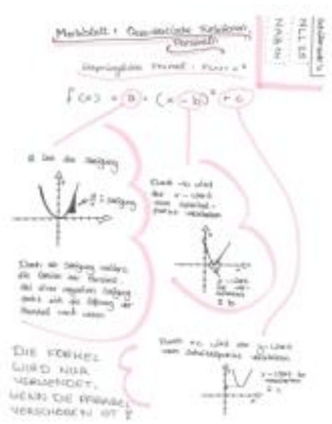


Figure 3: summary sheet example

A coding manual for the summary sheets was developed based on the qualitative content analysis of the summary sheets from two classes. Four main categories (language use, representation, structure, content) were set by the first and second named researchers and 84 sub-categories developed from the data. All summary sheets were coded by three master's thesis candidates after intensive instruction by the first named researcher. The codes for each summary sheet were assigned to each student in the group who produced the sheet. After the initial coding a first analysis was conducted and the results led to a refinement of the coding manual. During this refinement, some sub-categories were combined and some were coded in a slightly different way. After the revision of the coding manual the first named researcher then recoded all summary sheets. The results concerning the summary-sheets presented in this paper are based on the second round of coding.

RESULTS

Three different aspects of the results will be reported in this paper corresponding to the three sub-questions:

- technology-assisted guided discovery can support the learning of the concept of parameters,
- technology-assisted guided discovery can support students' exploration,
- technology-assisted guided discovery can support the testing of their hypotheses.

Additionally, a few constraints of the study will be represented.

Technology-assisted guided discovery can support learning of the concept of parameters

The following results are based on an analysis of the summary sheets. The analysis suggests that the dynamic visualisation using the "drag mode" and "sliders" supported students' investigation of the role of parameters more than static visualisation using the "function plotter". There are nearly no differences in the summary sheets results between the "function plotter" group and the "without visualisation" group. In nearly all of the sub-categories developed in the coding process of the summary sheets, the "drag mode" or "sliders" group gave the most *appropriate* answers, the "function plotter" and "without visualisation" group gave nearly always about the same amount of *appropriate* or *inappropriate* answers. In order to be classified as *appropriate* the responses on a summary sheet had to include ideas which were either already correct or which showed a preliminary or developing understanding.

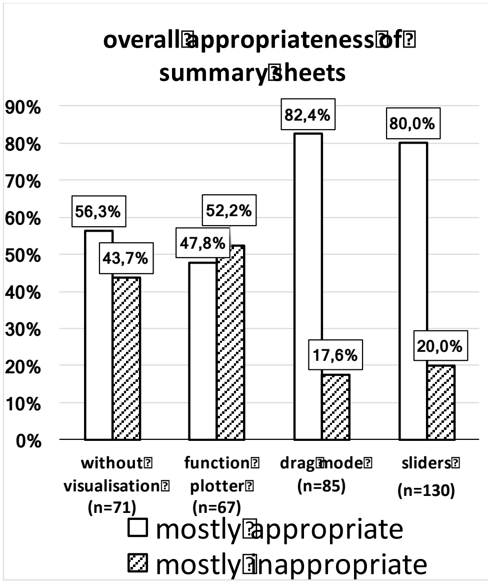


Figure 4: overall appropriateness of the summary sheets

The sub-category "Overall appropriateness" was used to classify the sheets according to the grade of correctness. Each sheet was considered as a whole and classified as *mostly appropriate* or *mostly inappropriate.* In order to be classified as *mostly appropriate* the summary sheets needed to contain more than 50% appropriate answers. The sheets of the "drag mode"- and the "sliders"-group gave *mostly appropriate* answers, whereas the "without visualisation"- and "function plotter" groups had approximately the same number of summary sheets in the two categories (refer to Figure 4). The dynamic visualisation through the use of "drag mode" and "sliders" seem to support development of students' knowledge of parameters more than by using static visualisation like function plotters or sketching the graphs by hand. This is evident through the ability of 80% or more of the students in the "drag mode"- and "sliders"-groups (refer to Figure 4) to provide statements classified *mostly appropriate*.

Another interesting point supporting the result that "drag mode"- and "sliders"-were more beneficial than a "function plotter" or no use of visualisation is that the only students who noted on their summary sheets that there is a special case when a=0 and the graph is a horizontal straight line, were in those two groups (i.e. "drag mode" or "sliders"). This special case (i.e. a=0) was recorded on the summary sheets of 37 students across the two groups, and of these 35 students noted this special case in an *appropriate* manner.

From the video and lesson observation notes, there was evidence that some students appeared to stumble across this special case by accident when using the technology to give negative values for the parameters; in moving from positive to negative values for the parameter 'a' students had to pass a=0, prompting them to further explore this case.

Technology-assisted guided discovery can support student exploration

The videos and lesson observations showed that in all three experimental groups and in the control group, students used a variety of different approaches to the task. Even though the task was pre-structured, a number of pairs in the "drag mode"- and "sliders"-groups explored their own relevant examples, rather than those provided, which still enabled them to explore the influence of the parameters using the iPads or handhelds.

For example, one pair of students (in the "sliders" group) initially followed the pre-structured task for describing the example of a transformed parabola, but three minutes after beginning their work they started exploring on their own, investigating examples which weren't suggested on the worksheet. This exploration was prompted when students were changing the values of the sliders to replicate the values in the example and they noted that changing parameters caused transformations. After doing this for a while (about 1 minute), the following statements were made (translated by the first-named author, text in italics describe the movements of the students).

TNA24: so a describes *(points with pen to slider for a)* this open and close, how wide it is open and how far it is closed *(IAR20 moves finger to slider b, then onto the equation)*

TNA24: b describes…ahm right or left *(moves pen to right and left, while IAR20 moves slider for b)*

IAR20: mh exactly…exactly

TNA24: c describes up or down, so move up. Yes but how can you, four is a *(writes something)*, isn't it?...but wait…

The transcript above is one example of how the students used the technology to explore and discover the influence of the different parameters in a very short time by manipulating the sliders to reproduce the given example. Concurrently, the students observed the change in the graph and hypothesized a generalisation for the effect of the parameter.

Technology-assisted guided discovery can support students to test their hypotheses

The same students used the technology to test their hypotheses. The students were working on Part 2 of the worksheet and as suggested they were investigating $f(x) = x^2 + c$ for different values for c. To do this they had put all sliders to zero first and then changed the one for c to 4.9. After zooming out so the graph was displayed the following statements were made (translated by the first-named author, italics describe movements of students).

TNA24: Oh didn't think that, ha but wait ahm, four point 9 (*points with pen on slider of c)* ah minus yes

IAR20: yes so when you, through c

TNA24: through c

IAR20: it is determined if it is a parabola, is the function

TNA24: a parabola, yes ok

IAR20 then changed the slider for c to 3.3 (refer to Figure 5) with stops in between around zero, while TNA24 states that it just moves, IAR20 replies:

IAR20: no wait, wait, no, c doesn't determine that, that must be something else, because otherwise it would have changed

TNA24: correct

IAR20: Then it has to be one of the other, so a or b

TNA24: so c only determines *(TNA24 points to the graph)*

IAR20: the y-intercept

They discuss for a minute about the vertex points of the graph and then change the slider for c back to zero and go on to the next part:

IAR20: now comes a *(moves slider for a)*

TNA24: a…that determines it, so a determines if it is a graph or not

IAR20: if it is a parabola or not

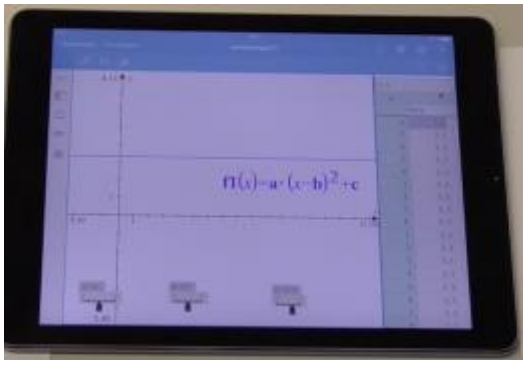


Figure 5: screenshot of the file the students are working on

They changed the slider for ‘a’ to zero and back again and then decided that ‘a’ determines if the graph is a parabola or not. These transcript passages show that the students used the technology to test and falsify their statement that c determines whether the graph is a parabola.

Limitations of the study

The videos, the summary sheets and the lesson observations show that a lot of the students were able to find out the effect on the graph, when one of the parameters is changed, but throughout all experimental groups and the control group reasons (correct or incorrect) for the transformations were not provided by most students. Some students tried arguing using knowledge of multiplication of fractions to explain the changes in the function tables when ‘a’ was changed, but they were largely unsuccessful in providing appropriate reasoning.

In order to support the students in finding explanations for the effect of changing ‘c’, there were two statements given in part 3 on the pre-structured worksheet concerning the shape of the graph, one of which was false. In addition, part 3 provided support for consideration of parameter ‘b’ by providing two correct statements asking students to reconstruct them. Unfortunately, this aspect of the pre-structured worksheet caused confusion for some students. When students realised that one of the statements for c was wrong they tried to find out which one of the two statements for parameter b was also wrong, rather than attempt to reconstruct the two correct statements as requested.

There were some technical difficulties which impacted students’ ability to focus on the underlying mathematical ideas. A number of the students in the “drag mode” group had difficulties with the accuracies of the given file. Due to the programming, the numbers in the function equation were displayed with two decimal places and the table with up to three decimal places. It was nearly impossible to get whole numbers as values for the parameters while dragging the graph. Hints from the teachers that the students should only try to achieve approximate values did not help much. The students lost considerable time trying to achieve exact values, which distracted them from the task. The difficulties achieving exact values made comparing values in different function tables very difficult.

A similar problem occurred in the “sliders”-group, where sliders could only be manipulated in 0.1 steps and the function tables value were displayed with two decimal places, resulting in students’ confusion again caused by accuracy of the displays. In addition, sliders were too small for some students to work with accurately and it took considerable time for some students to move sliders to obtain desired values. Despite this, it was observed that during the intervention lessons students tended to get more proficient with the sliders while working on the tasks, so the size issue could just be related to familiarity with the use of sliders.

DISCUSSION

The technology-assisted learning of the concepts of parameters in quadratic functions is possible in multiple ways, with all three different approaches having some potentials and constraints. Overall the "drag mode" seemed the most suitable to support the learning of the concept of parameter. Use of sliders was also found to support students' learning. The two experimental groups using "drag mode" and "sliders" were, on the whole, able to produce much more appropriate summary sheets than the control group, so it could be argued that the dynamic visualisation approaches have greater benefits for students' abilities to provide explanations for the effects of parameters on graphs than the static visualisation in the "function plotter" and control group. So in the frame of this study Zbiek et al.'s (2007) conjecture that the slider obscures rather than enhances students' understanding was not observed. On the contrary, the dynamic visualisation in the two groups "drag mode" and "sliders" proved to be beneficial for the investigation of the role of parameters. Even though this study involved a short intervention and did not explore the persistence of the learning gains, the results seemed promising. Overall, the study suggests that technology-assisted guided discovery is beneficial for the conceptualization of parameters.

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